A three-magnon problem for exactly rung-dimerized spin ladders: from a general outlook to the Bethe ansatz



2009 J. Phys. A: Math. Theor. 42 315212

(http://iopscience.iop.org/1751-8121/42/31/315212)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 315212 (29pp) doi:10.1088/1751-8113/42/31/315212

# A three-magnon problem for exactly rung-dimerized spin ladders: from a general outlook to the Bethe ansatz

**P N Bibikov**

Saint-Petersburg State University, 198504 Ulyanovskaya 1, Petrodvorets, Saint-Petersburg, Russia

Received 15 February 2009, in final form 24 May 2009
Published 16 July 2009
Online at stacks.iop.org/JPhysA/42/315212

**Abstract**
A three-magnon problem for an exactly rung-dimerized spin ladder is brought up separately in all total spin sectors. At first, a special duality transformation of the Schrödinger equation is found within the general outlook. Then the problem is treated within the coordinate Bethe ansatz. A straightforward approach is developed to obtain pure scattering states. At values $S = 0$ and $S = 3$ of total spin, the Schrödinger equation has a form inherent in the $XXZ$ chain. At $S = 1, 2$, solvability holds only in five previously found *completely integrable* cases. Nevertheless, even in a general non-integrable case, there are some special Bethe solutions in both $S = 1$ and $S = 2$ sectors. Pure scattering states in all total spin sectors are presented explicitly.

PACS number: 75.10.Pq

## 1. Introduction

Among other gapped 1D systems, spin ladders have been intensively studied during the last 15 years experimentally, numerically and theoretically (see references in [1–3]). This interest is accounted for by their possible relation to high temperature superconductivity, variety of static and dynamical properties and even the existence of several reliable compounds.

In the pioneering paper [4], a spin ladder was suggested as a double spin chain with only Heisenberg interactions both across and along the direction of the chains, namely rung and leg exchanges related to the couplings $J_{\rm r}$ and $J_{\rm l}$. It was also pointed that the case

$$J_{\rm r} \gg J_{\rm l} \tag{1}$$

has principle interest because it belongs to the so-called rung-dimerized phase in which almost all spins are coupled into rung singlets (rung dimers). In the purely Heisenberg model, this phase becomes exact only at $J_{\rm l} = 0$. However, it is always assumed that under condition (1), the physical picture does not change in general.

It soon became clear that the spin ladder Hamiltonian also presumes a term related to the diagonal Heisenberg coupling as well as four spin terms [5]. At first sight, these new interactions seemed to be complications for a theoretical analysis. However even in [6] it was noted that a special linear condition (equation (22) of this paper) on the former and new coupling constants guarantees for rather big $J_r$ (see estimations in [7]) the *exactness* of the rung-dimerized ground state. Moreover, in this case, all one- and two-magnon states may also be obtained in an explicit form [6, 7].

Unfortunately the rung-dimerization condition (22) has no reliable atomic level interpretation, so there is no physical reason to postulate it. Nevertheless it seems reasonable to suppose that for a strong rung exchange, any deviations from the exact rung-dimerized picture should be small and may be evaluated perturbatively. (This question will be studied in more detail in a forthcoming paper.) Under this point of view, an exactly rung-dimerized spin ladder is the best reference model for treating the whole rung-dimerized phase.

Some static and dynamic zero-temperature properties of exactly rung-dimerized spin ladders were studied in a series of papers [7–10]. Due to the existence of the gap, it succeeded to describe Raman scattering [7], magnetic phase transition [8] and (for asymmetric ladders) magnon decay [9, 10] utilizing only one- and two-magnon spectra. The latter problem was also studied by alternative approaches (see references in [10]). The three-magnon problem is less actual for the $T = 0$ physics (see, however, [11, 12] devoted to the $S = 1$ Haldane chain and $O(3)$ nonlinear $\sigma$-model).

Advancement into the $T > 0$ region requires knowledge of the whole spectrum [3, 16]. However, such a level of clarity may be achieved only for a rather limited list of the so-called integrable models [3, 13–17]. The latter are also significant in heat transport phenomena [18].

But how to find an integrable model? How can it be distinguished from a overwhelming majority of non-integrable ones? The most direct way is to express a treating Hamiltonian density as a derivative of the corresponding $R$-matrix which satisfies the Yang–Baxter equation. Solvability of this problem is governed by the Reshetikhin condition [17, 19, 20]. If the latter is satisfied for a given local Hamiltonian density, then the corresponding $R$-matrix very probably exists and may be obtained by an analysis of power series [20–22] or by some Yang–Baxterization ansatz [23]. In this paper, we suggest an alternative approach based on the solvability of the three-magnon problem in a framework of the coordinate Bethe ansatz (CBA)[1].

The essence of the CBA method [13, 14] is an assumption that any many-particle wavefunction is in fact a linear combination of terms produced by multiplications of one-particle exponents. Namely, for a rung-dimerized spin ladder the one-magnon wavefunction $\psi(n) = \mathrm{e}^{\mathrm{i}kn}$ [6] is parameterized by a real $0 \leqslant k < 2\pi$ (the wave number) and depends on an integer $n$ (position of the triplet rung). A two-magnon wavefunction $\psi(m,n)$ $(m < n)$ is a linear combination of two exponents $\mathrm{e}^{\mathrm{i}(k_1m+k_2n)}$ and $\mathrm{e}^{\mathrm{i}(k_2m+k_1n)}$ [7] and so depends on a pair of non-equal parameters $k_1$ and $k_2$. For a scattering state, both of them are real and one may put

$$0 \leqslant k_1 < k_2 < 2\pi, \tag{2}$$

while for a bound state they are a complex conjugate:

$$k_2 = \bar{k}_1. \tag{3}$$

In this light, it is seems reasonable to search for the representation of multi-magnon wavefunctions as sums of the Bethe exponents. However, even subsequent development of this approach to the three-magnon sector faces the problem of non-integrability.

[1] An idea to use a three-magnon problem as an alternative integrability test was privately prompted to the author by P P Kulish.

In order to reveal the origin of this obstacle, let us at first turn back to a two-magnon state. The total quasimomentum (wave number) and energy of the latter are represented by the sums

$$k = k_1 + k_2, \qquad E(k_1, k_2) = E_{\text{magn}}(k_1) + E_{\text{magn}}(k_2), \tag{4}$$

where $E_{\text{magn}}(k)$ is a single magnon energy. It is significant that under either condition (2) or condition (3), the mapping

$$k_1, k_2 \longrightarrow k, E \tag{5}$$

given by (4) is uniquely (up to an exchange $k_1 \leftrightarrow k_2$) reversible. However for three magnons, the situation is drastically different. Indeed, a system of relations

$$k = k_1 + k_2 + k_3, \qquad E = E_{\text{magn}}(k_1) + E_{\text{magn}}(k_2) + E_{\text{magn}}(k_3) \tag{6}$$

defines an infinite number of triples $(k_1, k_2, k_3)$. As a result, a three-magnon wavefunction related to the pair $(k, E)$ should contain in general an *infinite* number of exponential terms related to different solutions of system (6). Evidently, such a three-magnon problem is practically unsolvable.

The above obstacle may be overcome by the existence of a first integral (a translationary invariant operator commuting with the Hamiltonian) which produces the third condition additional to (6). An integrable system has an infinite number of such commuting with each other first integrals and may be solved in all multi-particle sectors. It is significant that within the CBA, a difference between integrability and non-integrability manifests just at the three-particle level. As a consequence of this fact, one may consider the solvability of the three-particle problem as an alternative integrability test.

In this paper, we study the three-magnon sector of a rung-dimerized symmetric spin ladder. At first, we briefly analyze the problem in general terms and only afterward do we turn to the CBA. Motivation behind such an approach lies in the following argumentation. Usually, the CBA is treated as a successful ad hoc conjecture which allows us to obtain in a rather straightforward manner all multi-particle states for a given quantum integrable model. However, the reference one is not integrable at general values of coupling constants. As a result (it will be shown below in detail), the CBA approach is applicable only in five special integrable cases.

The calculations are performed separately in the sectors $S = 0, 1, 2$ (the $S = 3$ sector is similar to the $S = 0$ one) of total spin. At $S = 0$ ($S = 3$), the system of equations on Bethe amplitudes has a well-known form inherent in the $XXZ$ spin chain and so is completely solvable for all values of coupling constants. For $S = 1$ and $S = 2$, a complete solvability exists only in the five integrable cases obtained earlier [21] within the Yang–Baxter framework. However even in the general non-integrable case there are some special (very complicated) solutions in both $S = 1$ and $S = 2$ sectors. Their interpretations remain unclear.

This paper is organized as follows. In section 2, we represent the spin ladder Hamiltonian in the most tractable form for which the rung-dimerized condition is evident. In section 3, we show that the Bethe form of the two-magnon wavefunction readily follows from a straightforward treatment of the Schrödinger equation. In section 4 treating the $S = 0$ ($S = 3$) sector within the general framework, we reveal a duality transformation of the wavefunction (generalized in sections 5 and 6 for $S = 1, 2$) and show that the Bethe ansatz readily follows from the factorized (Fourier) substitution. We also obtain a classification (generalized in sections 5 and 6 for $S = 1, 2$) of Bethe three-magnon states related to complex wave numbers. Pure scattering states obtained within the straightforward approach developed in sections 4–6 are presented in the appendices. In section 7, we show that the CBA solvability is in one-to-one correspondence with integrability. The latter was earlier revealed within the Yang–Baxter framework [21]. We also present the corresponding $R$-matrices. In section 8, within the CBA, we describe the action of the $S_3$ permutation group in all total spin sectors. This symmetry as well as duality described in sections 5 and 6 is used in the appendix for a more compact representation of Bethe states.

Since the ground state of the model has a simple factorized form, we treat it only in the infinite volume limit. An analogous approach to the ferromagnetic $XXZ$ chain was developed in [14].

## 2. The spin ladder Hamiltonian

Before presenting the spin ladder Hamiltonian, let us introduce the following local operators:

$$\begin{aligned}\mathbf{\Psi}_n &= \tfrac{1}{2}(\mathbf{S}_{1,n} - \mathbf{S}_{2,n}) - \mathrm{i}[\mathbf{S}_{1,n} \times \mathbf{S}_{2,n}],\\ \bar{\mathbf{\Psi}}_n &= \tfrac{1}{2}(\mathbf{S}_{1,n} - \mathbf{S}_{2,n}) + \mathrm{i}[\mathbf{S}_{1,n} \times \mathbf{S}_{2,n}]\end{aligned} \tag{7}$$

(we use the notation $\bar{\mathbf{\Psi}}_n$ instead of more convenient $\mathbf{\Psi}_n^*$ or $\mathbf{\Psi}_n^\dagger$ only in order to avoid such rather cumbersome notations as $(\mathbf{\Psi}_n^a)^*$). Here $\mathbf{S}_{1,n}$ and $\mathbf{S}_{2,n}$ are local spin operators associated with the $n$th rung. They may be expressed from $\mathbf{\Psi}_n$ and $\bar{\mathbf{\Psi}}_n$ as follows:

$$\begin{aligned}\mathbf{S}_{1,n} &= \tfrac{1}{2}(\mathbf{\Psi}_n + \bar{\mathbf{\Psi}}_n - \mathrm{i}[\bar{\mathbf{\Psi}}_n \times \mathbf{\Psi}_n]),\\ \mathbf{S}_{2,n} &= \tfrac{1}{2}(-\mathbf{\Psi}_n - \bar{\mathbf{\Psi}}_n - \mathrm{i}[\bar{\mathbf{\Psi}}_n \times \mathbf{\Psi}_n]).\end{aligned} \tag{8}$$

The representation (8) is similar to those suggested in [24, 25] but in fact it is not identical to them. Actually, the analogues of $\mathbf{\Psi}_n$ and $\bar{\mathbf{\Psi}}_n$ treated in [24, 25] act in an extended vector space. That is why, for example, the 'inverse' representation (7) fails for them. Operators (7) do not coincide with similar ones presented in [26, 27] although they act in the same vector space.

It may be readily proved that

$$[\mathbf{\Psi}_n, Q_n] = \mathbf{\Psi}_n, \qquad [\bar{\mathbf{\Psi}}_n, Q_n] = -\bar{\mathbf{\Psi}}_n, \tag{9}$$

where

$$Q_n = \tfrac{1}{2}\mathbf{S}_n^2, \qquad \mathbf{S}_n = \mathbf{S}_{1,n} + \mathbf{S}_{2,n}. \tag{10}$$

Let $|0\rangle_n$ and $|1\rangle_n$ be the correspondingly singlet and triplet states associated with the $n$th rung. From (10), it follows that

$$Q_n|0\rangle_n = 0, \qquad Q_n|1\rangle_n = |1\rangle_n, \tag{11}$$

so the local operator $Q_n$ is a projector on the $n$th rung triplet sector. Then according to equation (9) the two triples $\bar{\mathbf{\Psi}}_n$ and $\mathbf{\Psi}_n$ may be treated as rung-triplet creation–annihilation operators. Namely, a triple $|1\rangle_n^a$ $(a = x, y, z)$, for which

$$\bar{\mathbf{\Psi}}_n^a|0\rangle_n = |1\rangle_n^a, \qquad \bar{\mathbf{\Psi}}_n^a|1\rangle_n^b = 0, \qquad \mathbf{\Psi}_n^a|0\rangle_n = 0, \qquad \mathbf{\Psi}_n^a|1\rangle_n^b = \delta_{ab}|0\rangle_n, \tag{12}$$

gives the following representation of the total rung spin:

$$\mathbf{S}_n^a|1\rangle_n^b = \mathrm{i}\epsilon_{abc}|1\rangle_n^c \tag{13}$$

($\epsilon_{abc}$ is the Levi-Civita tensor). Parallel with (12), we shall use a triple

$$|1\rangle_n^j = \bar{\mathbf{\Psi}}_n^j|0\rangle_n, \qquad \mathbf{S}_n^z|1\rangle_n^j = j|1\rangle_n^j, \qquad j = -1, 0, 1, \tag{14}$$

related to operators

$$\bar{\mathbf{\Psi}}_n^{\pm 1} \equiv \tfrac{1}{\sqrt{2}}\left(\bar{\mathbf{\Psi}}_n^x \pm \mathrm{i}\bar{\mathbf{\Psi}}_n^y\right), \qquad \bar{\mathbf{\Psi}}_n^0 \equiv \bar{\mathbf{\Psi}}_n^z. \tag{15}$$

It seems reasonable to represent the Hamiltonian density $H_{n,n+1}$ for the general spin ladder Hamiltonian:

$$\hat{H} = \sum_n H_{n,n+1}, \tag{16}$$

in the following form:

$$\begin{aligned} H_{n,n+1} = {} & J_1(Q_n + Q_{n+1}) + J_2(\boldsymbol{\Psi}_n \cdot \bar{\boldsymbol{\Psi}}_{n+1} + \bar{\boldsymbol{\Psi}}_n \cdot \boldsymbol{\Psi}_{n+1}) + J_3 Q_n Q_{n+1} + J_4 \mathbf{S}_n \cdot \mathbf{S}_{n+1} \\ & + J_5(\mathbf{S}_n \cdot \mathbf{S}_{n+1})^2 + J_6(\bar{\boldsymbol{\Psi}}_n \cdot \bar{\boldsymbol{\Psi}}_{n+1} + \boldsymbol{\Psi}_n \cdot \boldsymbol{\Psi}_{n+1}). \end{aligned} \tag{17}$$

Up to a constant, this representation is equivalent to the standard one [1–6]

$$H_{n,n+1} = J_{\mathrm{r}} H^r_{n,n+1} + J_{\mathrm{l}} H^l_{n,n+1} + J_d H^d_{n,n+1} + J_{rr} H^{rr}_{n,n+1} + J_{ll} H^{ll}_{n,n+1} + J_{dd} H^{dd}_{n,n+1}, \tag{18}$$

where

$$\begin{aligned} H^r_{n,n+1} &= \tfrac{1}{2}(\mathbf{S}_{1,n} \cdot \mathbf{S}_{2,n} + \mathbf{S}_{1,n+1} \cdot \mathbf{S}_{2,n+1}), & H^l_{n,n+1} &= \mathbf{S}_{1,n} \cdot \mathbf{S}_{1,n+1} + \mathbf{S}_{2,n} \cdot \mathbf{S}_{2,n+1}, \\ H^d_{n,n+1} &= \mathbf{S}_{1,n} \cdot \mathbf{S}_{2,n+1} + \mathbf{S}_{2,n} \cdot \mathbf{S}_{1,n+1}, & H^{rr}_{n,n+1} &= (\mathbf{S}_{1,n} \cdot \mathbf{S}_{2,n})(\mathbf{S}_{1,n+1} \cdot \mathbf{S}_{2,n+1}), \\ H^{ll}_{n,n+1} &= (\mathbf{S}_{1,n} \cdot \mathbf{S}_{1,n+1})(\mathbf{S}_{2,n} \cdot \mathbf{S}_{2,n+1}), & H^{dd}_{n,n+1} &= (\mathbf{S}_{1,n} \cdot \mathbf{S}_{2,n+1})(\mathbf{S}_{2,n} \cdot \mathbf{S}_{1,n+1}), \end{aligned} \tag{19}$$

and

$$\begin{aligned} J_1 &= \tfrac{1}{4}(2J_{\mathrm{r}} - 3J_{rr} - J_{ll} - J_{dd}), & J_2 &= \tfrac{1}{8}(4(J_{\mathrm{l}} - J_d) + J_{ll} - J_{dd}), \\ J_3 &= J_{rr}, & J_4 &= \tfrac{1}{8}(4(J_{\mathrm{l}} + J_d) + J_{ll} + J_{dd}), \\ J_5 &= \tfrac{1}{4}(J_{ll} + J_{dd}), & J_6 &= \tfrac{1}{8}(4(J_{\mathrm{l}} - J_d) - J_{ll} + J_{dd}). \end{aligned} \tag{20}$$

It was suggested in [5] that only the case

$$J_{rr} = J_{ll} = -J_{dd} \tag{21}$$

(or equivalently $J_5 = 0$, $J_6 = J_2 - J_3/2$) has reliable interest. However since spin ladders with failed condition (21) are also currently studied [3], we shall not require it.

From (9) and (17), it directly follows that for

$$J_6 = 0 \quad \Leftrightarrow \quad J_{ll} - J_{dd} = 4(J_{\mathrm{l}} - J_d) \tag{22}$$

(creation and annihilation of rung-triplet pairs are suppressed), there holds

$$[\hat{H}, \hat{Q}] = 0. \tag{23}$$

Here the global operator

$$\hat{Q} = \sum_n Q_n, \tag{24}$$

according to (11), may be treated as a number operator for rung-triplets. The commutation relation (23) results in splitting of the Hilbert space into an infinite sum of eigenspaces related to different eigenvalues of $\hat{Q}$. In particular for rather strong $J_1$ (see estimations in [7]), the (zero energy) ground state of the model has a simple tensor-product form [6]

$$|0\rangle = \prod_n \otimes |0\rangle_n. \tag{25}$$

At the same time, the physical Hilbert space is subdivided into a direct sum of magnon sectors:

$$\mathcal{H} = \sum_{m=0}^{\infty} \mathcal{H}^m, \qquad \hat{Q}|_{\mathcal{H}^m} = m. \tag{26}$$

Only this special case (equation (22) and rather strong $J_1$) will be studied in this paper. Additionally, we shall imply that $J_2 \neq 0$. The completely diagonal frustrated model related to

$J_2 = 0$ or equivalently $J_d = J_1$ (in this case, the Hamiltonian density (17) may be expressed only in terms of $Q_n$ and $\mathbf{S}_n$) was studied in detail in [28]. Moreover, one may assume that

$$J_2 > 0 \quad \Leftrightarrow \quad J_1 > J_d. \tag{27}$$

Indeed, the case $J_2 < 0$ can be reduced to (27) by use of the following exchange of the coupling constants:

$$J_1 \leftrightarrow J_d, \qquad J_{ll} \leftrightarrow J_{dd}, \tag{28}$$

related to permutation of spins in all even (odd) rungs.

## 3. One- and two-magnon states

Taking into account (17), (11), (12) and (14), one gets the local formulae

$$\begin{aligned} H_{n,n+1}\cdots|1\rangle_n|0\rangle_{n+1}\cdots &= J_1\cdots|1\rangle_n|0\rangle_{n+1}\cdots + J_2\cdots|0\rangle_n|1\rangle_{n+1}\cdots,\\ H_{n-1,n}\cdots|0\rangle_{n-1}|1\rangle_n\cdots &= J_1\cdots|0\rangle_{n-1}|1\rangle_n\cdots + J_2\cdots|1\rangle_{n-1}|0\rangle_n\cdots \end{aligned} \tag{29}$$

and

$$\begin{aligned} &H_{n,n+1}\cdots|1\rangle_n^a|1\rangle_{n+1}^a\cdots = \varepsilon_0\cdots|1\rangle_n^a|1\rangle_{n+1}^a\cdots,\\ &H_{n,n+1}\epsilon_{abc}\cdots|1\rangle_n^b|1\rangle_{n+1}^c\cdots = \varepsilon_1\epsilon_{abc}\cdots|1\rangle_n^b|1\rangle_{n+1}^c\cdots, \qquad (a,b,c = x,y,z)\\ &H_{n,n+1}\cdots|1\rangle_n^+|1\rangle_{n+1}^+\cdots = \varepsilon_2\cdots|1\rangle_n^+|1\rangle_{n+1}^+\cdots. \end{aligned} \tag{30}$$

Here

$$\varepsilon_S \equiv 2(J_1 + J_2\Delta_S), \tag{31}$$

and

$$\begin{aligned} \Delta_0 &= \frac{J_3 - 2J_4 + 4J_5}{2J_2} = \frac{4(J_d - 2J_1) + 2J_{rr} + 3J_{ll}}{4(J_1 - J_d)},\\ \Delta_1 &= \frac{J_3 - J_4 + J_5}{2J_2} = \frac{4(J_{rr} - J_1) + J_{ll}}{8(J_1 - J_d)},\\ \Delta_2 &= \frac{J_3 + J_4 + J_5}{2J_2} = \frac{4(2J_d - J_1) + 4J_{rr} + 3J_{ll}}{8(J_1 - J_d)}. \end{aligned} \tag{32}$$

From (30), the following useful formula may be readily obtained:

$$\begin{aligned} H_{n,n+1}\cdots|1\rangle_n^a|1\rangle_{n+1}^b\cdots &= (2J_1 + J_3 + J_5)\cdots|1\rangle_n^a|1\rangle_{n+1}^b\cdots + J_4\cdots|1\rangle_n^b|1\rangle_{n+1}^a\cdots\\ &\quad + \delta_{ab}(J_5 - J_4)\cdots|1\rangle_n^c|1\rangle_{n+1}^c\cdots \qquad (a,b,c = x,y,z). \end{aligned} \tag{33}$$

Turning to excitation states, we note that an explicit form of a one-magnon state

$$|1,k\rangle = \sum_n \mathrm{e}^{ikn}\left(\prod_{m=-\infty}^{n-1}\otimes|0\rangle_m\right)\otimes|1\rangle_n\otimes\left(\prod_{m=n+1}^{\infty}\otimes|0\rangle_m\right) \tag{34}$$

directly follows from (23) and translation symmetry

$$\hat{P}|1,k\rangle = \mathrm{e}^{-ik}|1,k\rangle. \tag{35}$$

Here, $\hat{P}$ is the translation operator:

$$\hat{P}\prod\otimes|\chi(n)\rangle_n = \prod\otimes|\chi(n)\rangle_{n+1}, \qquad \chi(n) = 0, 1. \tag{36}$$

The corresponding dispersion

$$E_{\mathrm{magn}}(k) = 2(J_1 + J_2\cos k) \tag{37}$$

readily follows from (29).

Since the $\hat{Q} = 2$ sector is subdivided into $S = 0, 1, 2$ total spin subsectors, we denote at once a two-magnon state with total spin $S$ and wave vector $k$ as $|2, S, k\rangle$. The following general representations for the two-magnon states

$$
\begin{aligned}
|2,0,k\rangle &= \sum_{m<n} \mathrm{e}^{\mathrm{i}k(m+n)/2} a_0(k, n-m) \cdots |1\rangle_m^a \cdots |1\rangle_n^a \cdots, \\
|2,1,k\rangle^a &= \varepsilon_{abc} \sum_{m<n} \mathrm{e}^{\mathrm{i}k(m+n)/2} a_1(k, n-m) \cdots |1\rangle_m^b \cdots |1\rangle_n^c \cdots, \\
|2,2,k\rangle^{+2} &= \sum_{m<n} \mathrm{e}^{\mathrm{i}k(m+n)/2} a_2(k, n-m) \cdots |1\rangle_m^+ \cdots |1\rangle_n^+ \cdots
\end{aligned}
\tag{38}
$$

agree with the rotational and translational (35) symmetries. From equation (38), by '$\cdots$', we denote an appropriate tensor product of *rung singlets* (similar to products in (34)). For simplicity, the $S = 2$ sector is represented in (38) by the $\mathbf{S}^z = +2$ states. In addition, we suggest that the reduced wavefunction $a_S(k, n)$ should be bounded

$$
\sup_n a_S(k, n) < \infty. \tag{39}
$$

The Schrödinger equation for $a_S(k, n)$ has different forms at $n > 1$ and $n = 1$. In the former case, equations (29) and (30) give

$$
4J_1 a_S(k, n) + 2J_2 \cos\frac{k}{2}[a_S(k, n-1) + a_S(k, n+1)] = E a_S(k, n), \tag{40}
$$

while in the latter

$$
(2J_1 + \varepsilon_S) a_S(k, 1) + 2J_2 \cos\frac{k}{2} a_S(k, 2) = E(k) a_S(k, 1). \tag{41}
$$

It is convenient to rewrite equation (41) in the form of equation (40) [13] by continuing $a_S(k, n)$ into an unphysical region $n = 0$. Comparing (40) and (41), one concludes that this trick entails the following Bethe condition:

$$
\Delta_S a_S(k, 1) = \cos\frac{k}{2} a_S(k, 0). \tag{42}
$$

System (40) (now considered for $n \geqslant 1$) and (42) together with the restriction (39) allows us to obtain entire $a_S(k, n)$ in a straightforward manner. Indeed representing equation (40) in an equivalent matrix form

$$
\begin{pmatrix} a_S(k, n+1) \\ a_S(k, n) \end{pmatrix} = \mathcal{F}(\kappa) \begin{pmatrix} a_S(k, n) \\ a_S(k, n-1) \end{pmatrix}, \tag{43}
$$

where

$$
\mathcal{F}(\kappa) = \begin{pmatrix} 2\kappa & -1 \\ 1 & 0 \end{pmatrix}, \qquad \kappa = \frac{E - 4J_1}{4J_2 \cos k/2}, \tag{44}
$$

and taking $a_S(k, 1) : a_S(k, 0)$ from (42), one consequently obtains (up to a constant factor) the rest of $a_S(k, n)$ at $n = 2, 3, \ldots$ by using (43). In the following, we shall study this problem in detail by separately considering three regions $|\kappa| < 1$, $|\kappa| > 1$ and $|\kappa| = 1$.

For $|\kappa| \neq 1$, the matrix $\mathcal{F}(\kappa)$ has two different eigenvalues

$$
\Lambda_\pm(\kappa) = \kappa \pm \sqrt{\kappa^2 - 1}, \tag{45}
$$

related to eigenvectors

$$
\xi_\pm(\kappa) = \begin{pmatrix} \Lambda_\pm(\kappa) \\ 1 \end{pmatrix}. \tag{46}
$$

At $|\kappa| < 1$, it is more convenient to use the following representation:

$$\Lambda_{\pm}(\kappa) = \mathrm{e}^{\pm iq}, \qquad \kappa = \cos q, \qquad 0 < q < \pi. \tag{47}$$

According to (43), a decomposition

$$\begin{pmatrix} a_S(k,1) \\ a_S(k,0) \end{pmatrix} = c_+\xi_+(\kappa) + c_-\xi_-(\kappa), \tag{48}$$

($c_{\pm}$ are some coefficients) results in

$$\begin{pmatrix} a_S(k,n+1) \\ a_S(k,n) \end{pmatrix} = \Lambda_+^n(\kappa)c_+\xi_+(\kappa) + \Lambda_-^n(\kappa)c_-\xi_-(\kappa) \tag{49}$$

or equivalently

$$a_S^{\mathrm{scatt}}(k,q,n) = \cos\frac{k}{2}\sin qn - \Delta_S \sin q(n-1). \tag{50}$$

Expression (50) obviously agrees with (39) and (as it readily follows from (29) and (30)) corresponds to dispersion

$$E_{\mathrm{scatt}}(k,q) = 4\left(J_1 + J_2\cos q\cos\frac{k}{2}\right). \tag{51}$$

According to the following formulae:

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}k(m+n)/2}a_S^{\mathrm{scatt}}(k,q,n-m) = \frac{1}{2\mathrm{i}}[C_{S,12}\,\mathrm{e}^{\mathrm{i}(k_1m+k_2n)} - C_{S,21}\,\mathrm{e}^{\mathrm{i}(k_2m+k_1n)}],\\ &E_{\mathrm{scatt}}(k,q) = E_{\mathrm{magn}}(k_1) + E_{\mathrm{magn}}(k_2), \end{aligned} \tag{52}$$

where

$$\frac{k}{2} - q = k_1 < k_2 = \frac{k}{2} + q, \qquad q = \frac{k_2 - k_1}{2}, \tag{53}$$

and

$$C_{S,ab} = \cos\frac{k_a + k_b}{2} - \Delta_S\,\mathrm{e}^{\mathrm{i}(k_a - k_b)/2}, \tag{54}$$

one may associate (50) with a scattering wavefunction of two magnons with wave vectors $k_1$ and $k_2$ reduced to the center-of-mass frame.

For $|\kappa| > 1$, both the eigenvalues (45) are real. More specifically, at $\pm\kappa > 1$ there should be $|\Lambda_{\mp}(\kappa)| < 1 < |\Lambda_{\pm}(\kappa)|$ and the representation (49) agrees with (39) only for $c_{\pm} = 0$. According to (42), (46) and (48) in both these cases, the remaining eigenvalue in (49) is $(\cos k/2)/\Delta_S$. So one gets

$$\begin{aligned} &a_S^{\mathrm{bound}}(k,n) = \left(\frac{\cos k/2}{\Delta_S}\right)^n,\\ &E_{\mathrm{bound}}(S,k) = 2\left(2J_1 + J_2\Delta_S + \frac{J_2}{\Delta_S}\cos^2\frac{k}{2}\right). \end{aligned} \tag{55}$$

This solution exists only for

$$-|\Delta_S| < \cos\frac{k}{2} < |\Delta_S|, \tag{56}$$

and according to the formulae

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}k(m+n)/2}a_S^{\mathrm{bound}}(k,n-m) = \mathrm{e}^{\mathrm{i}(k_1m+k_2n)},\\ &E_{\mathrm{bound}}(S,k) = E_{\mathrm{magn}}(k_1) + E_{\mathrm{magn}}(k_2), \end{aligned} \tag{57}$$

where

$$k_1 = \frac{k}{2} - \mathrm{i}v, \qquad k_2 = \frac{k}{2} + \mathrm{i}v, \qquad v = \ln\left(\frac{\Delta_S}{\cos k/2}\right), \tag{58}$$

it may be associated with a two-magnon bound-state wavefunction reduced to the center-of-mass frame. Both (52) and (57) reproduce the Bethe ansatz calculation presented in [7].

For $\kappa^2 = 1$, the matrix $\mathcal{F}(\kappa)$ has an eigenvector $\xi_0$ and an adjoint vector $\tilde{\xi}_0$:

$$\xi_0 = \begin{pmatrix}\kappa\\ 1\end{pmatrix}, \qquad \tilde{\xi}_0 = \begin{pmatrix}1\\ 0\end{pmatrix}. \tag{59}$$

There holds

$$\mathcal{F}(\kappa)\xi_0(\kappa) = \kappa\xi_0(\kappa), \qquad \mathcal{F}(\kappa)\tilde{\xi}_0(\kappa) = \kappa\tilde{\xi}_0(\kappa) + \xi_0(\kappa). \tag{60}$$

Taking into account (60) and (42), one gets the following decomposition:

$$\begin{pmatrix}a_S(k,1)\\ a_S(k,0)\end{pmatrix} = \Delta_S\xi_0(\kappa) + \left(\cos\frac{k}{2} - \kappa\Delta_S\right)\tilde{\xi}_0(\kappa). \tag{61}$$

The resulting wavefunction

$$a_S(k,\kappa,n) = n\kappa^{n-1}\left(\cos\frac{k}{2} - \kappa\Delta_S\right) + \kappa^n\Delta_S \tag{62}$$

agrees with (39) only on the appropriate bound of interval (56), namely for

$$\cos\frac{k}{2} = \kappa\Delta_S. \tag{63}$$

Solution (62) may be obtained from both (50) and (55) in the limit $|\kappa| \to 1$. Indeed despite the fact that wavefunction (50) turns to zero at $q = 0, \pi$, the ratio $a^{\mathrm{scatt}}(k,q,n)/\sin q$ remains finite and gives (62) as a limit value. Analogously, using the formula $(1+\epsilon)^n = 1 + n\epsilon + o(\epsilon)$, one can obtain (62) from (55).

## 4. $S = 0$ and $S = 3$ three-magnon sectors

Representing at once a $S = 0$ state in a general translationary covariant form

$$|3,0,k\rangle = \epsilon_{abc}\sum_{m<n<p}\mathrm{e}^{\mathrm{i}k(m+n+p)/3}b_0(k,n-m,p-n)\cdots|1\rangle_m^a\cdots|1\rangle_n^b\cdots|1\rangle_p^c\cdots, \tag{64}$$

one readily obtains from (29) and (33) a Schrödinger equation for the reduced wavefunction $b_0(k,m,n)$. In the $m, n > 1$ sector

$$\begin{aligned}6J_1b_0(k,m,n) &+ J_2[\mathrm{e}^{-\mathrm{i}k/3}b_0(k,m+1,n) + \mathrm{e}^{\mathrm{i}k/3}b_0(k,m-1,n) + \mathrm{e}^{-\mathrm{i}k/3}b_0(k,m-1,n+1)\\ &+ \mathrm{e}^{\mathrm{i}k/3}b_0(k,m+1,n-1) + \mathrm{e}^{-\mathrm{i}k/3}b_0(k,m,n-1) + \mathrm{e}^{\mathrm{i}k/3}b_0(k,m,n+1)]\\ &= Eb_0(k,m,n),\end{aligned} \tag{65}$$

while for $m, n = 1$

$$\begin{aligned}(4J_1+\varepsilon_1)b_0(k,1,n) &+ J_2[\mathrm{e}^{-\mathrm{i}k/3}b_0(k,1,n-1) + \mathrm{e}^{\mathrm{i}k/3}b_0(k,1,n+1)\\ &+ \mathrm{e}^{\mathrm{i}k/3}b_0(k,2,n-1) + \mathrm{e}^{-\mathrm{i}k/3}b_0(k,2,n)] = Eb_0(k,1,n),\\ (4J_1+\varepsilon_1)b_0(k,m,1) &+ J_2[\mathrm{e}^{\mathrm{i}k/3}b_0(k,m-1,1) + \mathrm{e}^{-\mathrm{i}k/3}b_0(k,m+1,1)\\ &+ \mathrm{e}^{-\mathrm{i}k/3}b_0(k,m-1,2) + \mathrm{e}^{\mathrm{i}k/3}b_0(k,m,2)] = Eb_0(k,m,1).\end{aligned} \tag{66}$$

Reduction of (66) to (65) results in a system of Bethe conditions,

$$\begin{aligned}2\Delta_1b_0(k,1,n) &= \mathrm{e}^{\mathrm{i}k/3}b_0(k,0,n) + \mathrm{e}^{-\mathrm{i}k/3}b_0(k,0,n+1),\\ 2\Delta_1b_0(k,m,1) &= \mathrm{e}^{-\mathrm{i}k/3}b_0(k,m,0) + \mathrm{e}^{\mathrm{i}k/3}b_0(k,m+1,0).\end{aligned} \tag{67}$$

The pair (65) (considered for $m, n > 0$) and (67) which represents the Schrödinger equation on $b_0(k, m, n)$ is invariant under the following duality transformation:

$$\mathcal{D}(b_0(k, m, n)) = \bar{b}_0(k, n, m). \tag{68}$$

Multiplication on $i$ transforms autodual and anti-autodual solutions from one to another.

As in the two-magnon case, we suggest that the reduced wavefunction should be bounded:

$$\sup_{m,n} b_0(k, m, n) < \infty. \tag{69}$$

Despite the fact that system (65), (67) is linear, a proper generalization of the straightforward matrix approach used in the previous section is unclear for it. Instead, one may treat (65) by Fourier substitution

$$\tilde{b}_0(k, m, n) = \varphi(k, m)\theta(k, n), \tag{70}$$

which results in the following two-parametric exponential solution:

$$\tilde{b}_0(k, m, n) = \mathrm{e}^{\mathrm{i}(\tilde{q}_1 m + \tilde{q}_2 n)}, \tag{71}$$

related to dispersion

$$E(k, \tilde{q}_1, \tilde{q}_2) = \sum_{j=1}^{3} E_{\mathrm{magn}}(k_j), \tag{72}$$

where

$$k_1 = \frac{k}{3} - \tilde{q}_1, \qquad k_2 = \frac{k}{3} + \tilde{q}_1 - \tilde{q}_2, \qquad k_3 = \frac{k}{3} + \tilde{q}_2. \tag{73}$$

Since

$$\mathrm{e}^{\mathrm{i}k/3(m+n+p)}\tilde{b}_0(k, n-m, p-n) = \mathrm{e}^{\mathrm{i}(k_1 m + k_2 n + k_3 p)}, \tag{74}$$

one can naturally associate (71) with the wavefunction of a magnon triple with wave numbers $k_1, k_2, k_3$.

Instead of $\tilde{q}_{1,2}$, we shall mainly use the parameters

$$q_1 = \frac{k_2 - k_1}{2} = \tilde{q}_1 - \frac{\tilde{q}_2}{2}, \qquad q_2 = \frac{k_3 - k_2}{2} = \tilde{q}_2 - \frac{\tilde{q}_1}{2}, \tag{75}$$

considering them as generalizations of the parameter $q$ from equation (53). The pair $q_{1,2}$ is more convenient for representation of *pure scattering* states related to real $k_{1,2,3}$ with $0 \leqslant k_1 < k_2 < k_3 \leqslant 2\pi$ (a generalization of equation (2)) because the latter system of inequalities in terms of $q_{1,2}$ has a very simple form. Namely $0 < q_{1,2} < \pi$ and $0 < q_1 + q_2 < \pi$. Nevertheless due to a rather compact representation (71) the parameters $\tilde{q}_{1,2}$ will still remain in some exponential factors. They will also be used for classification of states with complex wave numbers (equation (82)).

For complex $\tilde{q}_{1,2}$, one has to carefully treat condition (69) and to take into account the fact that the energy (72) must be real. These conditions result in

$$\mathrm{Im}(\tilde{q}_j) \geqslant 0, \qquad j = 1, 2, \tag{76}$$

$$\begin{aligned}(\sin[k/3 - \mathrm{Re}(\tilde{q}_1)] &- \sin[k/3 + \mathrm{Re}(\tilde{q}_1 - \tilde{q}_2)]\cosh \mathrm{Im}(\tilde{q}_2))\sinh \mathrm{Im}(\tilde{q}_1)\\ &= (\sin[k/3 + \mathrm{Re}(\tilde{q}_2)] - \sin[k/3 + \mathrm{Re}(\tilde{q}_1 - \tilde{q}_2)]\cosh \mathrm{Im}(\tilde{q}_1))\sinh \mathrm{Im}(\tilde{q}_2).\end{aligned} \tag{77}$$

The dispersion (72) is invariant under permutations of $k_1$, $k_2$ and $k_3$ or equivalently under the following transformations of $\tilde{\boldsymbol{q}} \equiv (\tilde{q}_1, \tilde{q}_2)$:

$$\begin{aligned}&\omega_1(\tilde{\boldsymbol{q}}) = (\tilde{q}_1, \tilde{q}_1 - \tilde{q}_2), \qquad \omega_2(\tilde{\boldsymbol{q}}) = (-\tilde{q}_2, \tilde{q}_1 - \tilde{q}_2), \qquad \omega_3(\tilde{\boldsymbol{q}}) = (-\tilde{q}_2, -\tilde{q}_1)\\ &\omega_4(\tilde{\boldsymbol{q}}) = (\tilde{q}_2 - \tilde{q}_1, -\tilde{q}_1), \qquad \omega_5(\tilde{\boldsymbol{q}}) = (\tilde{q}_2 - \tilde{q}_1, \tilde{q}_2).\end{aligned} \tag{78}$$

In fact, these formulae give a representation of the three-element permutation group $S_3$. It may be readily checked that all $\omega_j$ are generated by $\omega_1$ and $\omega_5$. Namely,

$$\omega_2 = \omega_5 \cdot \omega_1, \qquad \omega_3 = \omega_5 \cdot \omega_1 \cdot \omega_5 = \omega_1 \cdot \omega_5 \cdot \omega_1, \qquad \omega_4 = \omega_1 \cdot \omega_5. \tag{79}$$

The symmetry (78) allows us to generalize solution (71) and suggests the following ansatz ($\mathbf{q} \equiv (q_1, q_2)$):

$$\begin{aligned} b_0(k, \mathbf{q}, m, n) = {} & A_1(k, \mathbf{q})\, e^{i(\tilde{q}_1 m + \tilde{q}_2 n)} - A_2(k, \mathbf{q})\, e^{i(\tilde{q}_1 m + (\tilde{q}_1 - \tilde{q}_2) n)} + A_3(k, \mathbf{q})\, e^{i(-\tilde{q}_2 m + (\tilde{q}_1 - \tilde{q}_2) n)} \\ & - A_4(k, \mathbf{q})\, e^{-i(\tilde{q}_2 m + \tilde{q}_1 n)} + A_5(k, \mathbf{q})\, e^{i((\tilde{q}_2 - \tilde{q}_1) m - \tilde{q}_1 n)} - A_6(k, \mathbf{q})\, e^{i((\tilde{q}_2 - \tilde{q}_1) m + \tilde{q}_2 n)}. \end{aligned} \tag{80}$$

For $\mathrm{Im}(\tilde{q}_{1,2}) = 0$, expression (80) agrees with (69) while equation (77) is satisfied identically. However even when one of $\tilde{q}_{1,2}$ has an imaginary part some of the amplitudes in (80) must turn to zero in order to ensure an agreement with (69). Additionally according to (77), the real and imaginary parts of $\tilde{q}_{1,2}$ should be interdependent. More specifically let us divide the sector $\mathrm{Im}(\tilde{q}_{1,2}) \geqslant 0$ into five subsectors:

$$\begin{aligned} &\mathcal{V}_1 = [\mathrm{Im}(\tilde{q}_1) > 0, \mathrm{Im}(\tilde{q}_2) = 0], \qquad \mathcal{V}_2 = [\mathrm{Im}(\tilde{q}_1) = 0, \mathrm{Im}(\tilde{q}_2) > 0], \\ &\mathcal{V}_3 = [0 < \mathrm{Im}(\tilde{q}_1) = \mathrm{Im}(\tilde{q}_2)], \qquad \mathcal{V}_4 = [0 < \mathrm{Im}(\tilde{q}_1) < \mathrm{Im}(\tilde{q}_2)], \\ &\mathcal{V}_5 = [0 < \mathrm{Im}(\tilde{q}_2) < \mathrm{Im}(\tilde{q}_1)]. \end{aligned} \tag{81}$$

For each $\mathcal{V}_i$, let $\mathcal{J}_i$ be the corresponding set of $l$'s for which there should be $A_l(k, \mathbf{q}) = 0$. At the same time, $\mathcal{Q}_i$ will be the corresponding additional condition on $\tilde{q}_{1,2}$ following from (77). For each $i$, we may define a triple $\mathcal{W}_i = [\mathcal{V}_i; \mathcal{J}_i; \mathcal{G}_i]$. A straightforward analysis based on equations (77) and (80) results in the following classification:

$$\begin{aligned} &\mathcal{W}_1 = [\mathrm{Im}(\tilde{q}_1) > 0, \mathrm{Im}(\tilde{q}_2) = 0; \{4, 5, 6\}; \mathrm{Re}(\tilde{q}_2) = 2\,\mathrm{Re}(\tilde{q}_1)], \\ &\tilde{\mathcal{W}}_1 = [\mathrm{Im}(\tilde{q}_1) > 0, \mathrm{Im}(\tilde{q}_2) = 0; \{4, 5, 6\}; \mathrm{Re}(\tilde{q}_2) = 2k/3 + \pi], \\ &\mathcal{W}_2 = [\mathrm{Im}(\tilde{q}_1) = 0, \mathrm{Im}(\tilde{q}_2) > 0; \{2, 3, 4\}; \mathrm{Re}(\tilde{q}_1) = 2\,\mathrm{Re}(\tilde{q}_2)], \\ &\tilde{\mathcal{W}}_2 = [\mathrm{Im}(\tilde{q}_1) = 0, \mathrm{Im}(\tilde{q}_2) > 0; \{2, 3, 4\}; \mathrm{Re}(\tilde{q}_1) = -2k/3 + \pi], \\ &\mathcal{W}_3 = [0 < \mathrm{Im}(\tilde{q}_1) = \mathrm{Im}(\tilde{q}_2); \{3, 4, 5\}; \mathrm{Re}(\tilde{q}_2) = -\mathrm{Re}(\tilde{q}_1)], \\ &\tilde{\mathcal{W}}_3 = [0 < \mathrm{Im}(\tilde{q}_1) = \mathrm{Im}(\tilde{q}_2); \{3, 4, 5\}; \mathrm{Re}(\tilde{q}_2) = \mathrm{Re}(\tilde{q}_1) + \pi - 2k/3], \\ &\mathcal{W}_4 = [0 < \mathrm{Im}(\tilde{q}_1) < \mathrm{Im}(\tilde{q}_2); \{2, 3, 4, 5\}; \text{equation (77)}], \\ &\mathcal{W}_5 = [0 < \mathrm{Im}(\tilde{q}_2) < \mathrm{Im}(\tilde{q}_1); \{3, 4, 5, 6\}; \text{equation (77)}]. \end{aligned} \tag{82}$$

Each state with complex $\tilde{q}$-s corresponds without fail to one of the $\mathcal{W}$-s presented in (82).

System (65) (at $m, n \geqslant 1$), (67) exactly coincides with the well-known one inherent in the $XXZ$ model [13, 14]. Nevertheless, we shall give its solution within the ansatz (80) in order to illustrate the straightforward approach used in the following section for $S = 1$ and $S = 2$.

Let us begin with pure scattering states for which the duality transformation (68) results in

$$\mathcal{D}(A_l(k, \mathbf{q})) = -\bar{A}_{l-3}(k, \mathbf{q}), \tag{83}$$

where $A_l(k, \mathbf{q}) \equiv A_{l+6}(k, \mathbf{q})$ for $l = -2, -1, 0$. The $S = 1$ and $S = 2$ analogues of this formula will be used in the appendix for enumeration of three-magnon Bethe states.

Substitution of (80) into (67) produces a linear system on the amplitudes $A_l(k, \mathbf{q})$:

$$\sum_{l=1}^{6} M_{il}^{(0)}(k, \mathbf{q}) A_l(k, \mathbf{q}) = 0, \tag{84}$$

where nonzero entries of the $6 \times 6$ matrix $M^{(0)}(k, \mathbf{q})$ are

$$\begin{aligned}
M^{(0)}_{11}(k, \mathbf{q}) &= -M^{(0)}_{45}(k, \mathbf{q}) = Z(k, \omega_5(\mathbf{q}), \Delta_1),\\
M^{(0)}_{22}(k, \mathbf{q}) &= -M^{(0)}_{56}(k, \mathbf{q}) = Z(k, \mathbf{q}, \Delta_1),\\
M^{(0)}_{33}(k, \mathbf{q}) &= -M^{(0)}_{61}(k, \mathbf{q}) = Z(k, \omega_1(\mathbf{q}), \Delta_1),\\
M^{(0)}_{44}(k, \mathbf{q}) &= -M^{(0)}_{12}(k, \mathbf{q}) = Z(k, \omega_2(\mathbf{q}), \Delta_1),\\
M^{(0)}_{55}(k, \mathbf{q}) &= -M^{(0)}_{23}(k, \mathbf{q}) = Z(k, \omega_3(\mathbf{q}), \Delta_1),\\
M^{(0)}_{66}(k, \mathbf{q}) &= -M^{(0)}_{34}(k, \mathbf{q}) = Z(k, \omega_4(\mathbf{q}), \Delta_1).
\end{aligned} \tag{85}$$

Here

$$Z(k, \mathbf{q}, \Delta) = \cos\left(\frac{k + q_2 - q_1}{3}\right) - \Delta\, \mathrm{e}^{\mathrm{i}(q_1+q_2)}, \tag{86}$$

while according to (75) and (78)

$$\begin{aligned}
&\omega_1(\mathbf{q}) = (q_1 + q_2, -q_2), \qquad \omega_2(\mathbf{q}) = (-q_1 - q_2, q_1), \qquad \omega_3(\mathbf{q}) = (-q_2, -q_1),\\
&\omega_4(\mathbf{q}) = (q_2, -q_1 - q_2), \qquad \omega_5(\mathbf{q}) = (-q_1, q_1 + q_2).
\end{aligned} \tag{87}$$

Since

$$\det M^{(0)}(k, \mathbf{q}) = \prod_{n=1..6} M^{(0)}_{nn}(k, \mathbf{q}) - \prod_{n=1..6} M^{(0)}_{n,n+1}(k, \mathbf{q}) = 0 \tag{88}$$

(here $M^{(0)}_{67}(k, \mathbf{q}) \equiv M^{(0)}_{61}(k, \mathbf{q})$), the matrix system (84) is solvable. Namely

$$A_l(k, \mathbf{q}) = \prod_{i=1}^{3} M^{(0)}_{l-i,l-i}(k, \mathbf{q}), \tag{89}$$

where $M^{(0)}_{ll}(k, \mathbf{q}) \equiv M^{(0)}_{l+6,l+6}(k, \mathbf{q})$ for $l = -2, -1, 0$.

States with complex $\tilde{q}_{1,2}$ may be obtained from (89) by analytic continuation with regard to conditions presented in list (82). It may be readily shown by straightforward calculations that there are no solutions related to $\tilde{\mathcal{W}}_{1,2,3}$ and $\mathcal{W}_{4,5}$. This statement is a special (related to the three-magnon sector) confirmation of the string hypothesis proved for the $XXZ$ chain [13, 14].

It may be readily proved that the $S = 3$ case is analogous to the $S = 0$ one. It is only necessary to improve the representation (64) (in order to obtain the state with total spin $S = 3$) and to replace $\Delta_1$ with $\Delta_2$ everywhere.

## 5. $S$ = 1 three-magnon sector

A general $S = 1$ three-magnon state has the following representation:

$$\begin{aligned}
|3, 1, k\rangle^a = \sum_{m<n<p} \mathrm{e}^{\mathrm{i}k(m+n+p)/3}\big[&b^{(1)}_1(k, n-m, p-n)\cdots|1\rangle^a_m\cdots|1\rangle^b_n\cdots|1\rangle^b_p\cdots\\
&+ b^{(2)}_1(k, n-m, p-n)\cdots|1\rangle^b_m\cdots|1\rangle^a_n\cdots|1\rangle^b_p\cdots\\
&+ b^{(3)}_1(k, n-m, p-n)\cdots|1\rangle^b_m\cdots|1\rangle^b_n\cdots|1\rangle^a_p\cdots\big],
\end{aligned} \tag{90}$$

and depends on the three wavefunctions $b^{(1,2,3)}_1(k, m, n)$. At $m, n > 1$, the Schrödinger equation for $b^{(1,2,3)}_1(k, m, n)$ separates into three independent linear subsystems of form (65) (one have only to replace $b_0(k,m,n)$ with $b_1^{(1,2,3)}(k,m,n)$). However for $m,n=1$, one gets

$$
\begin{aligned}
&\left(6J_1+J_2+\tfrac{3}{2}J_3\right)b_1^{(1)}(k,1,n)+J_4b_1^{(2)}(k,1,n)+J_2\big[\mathrm{e}^{-\mathrm{i}k/3}b_1^{(1)}(k,1,n-1)+\mathrm{e}^{\mathrm{i}k/3}b_1^{(1)}(k,1,n+1)\\
&\qquad+\mathrm{e}^{\mathrm{i}k/3}b_1^{(1)}(k,2,n-1)+\mathrm{e}^{-\mathrm{i}k/3}b_1^{(1)}(k,2,n)\big]=Eb_1^{(1)}(k,1,n),\\
&\left(6J_1+J_2+\tfrac{3}{2}J_3\right)b_1^{(2)}(k,1,n)+J_4b_1^{(1)}(k,1,n)+J_2\big[\mathrm{e}^{-\mathrm{i}k/3}b_1^{(2)}(k,1,n-1)\\
&\qquad+\mathrm{e}^{\mathrm{i}k/3}b_1^{(2)}(k,1,n+1)+\mathrm{e}^{\mathrm{i}k/3}b_1^{(2)}(k,2,n-1)+\mathrm{e}^{-\mathrm{i}k/3}b_1^{(2)}(k,2,n)\big]\\
&\quad=Eb_1^{(2)}(k,1,n),\\
&(4J_1+\varepsilon_0)b_1^{(3)}(k,1,n)+(J_5-J_4)\big(b_1^{(1)}(k,1,n)+b_1^{(2)}(k,1,n)\big)\\
&\qquad+J_2\big[\mathrm{e}^{-\mathrm{i}k/3}b_1^{(3)}(k,1,n-1)+\mathrm{e}^{\mathrm{i}k/3}b_1^{(3)}(k,1,n+1)+\mathrm{e}^{\mathrm{i}k/3}b_1^{(3)}(k,2,n-1)\\
&\qquad+\mathrm{e}^{-\mathrm{i}k/3}b_1^{(3)}(k,2,n)\big]=Eb_1^{(3)}(k,1,n),\\
&(4J_1+\varepsilon_0)b_1^{(1)}(k,m,1)+(J_5-J_4)\big[b_1^{(2)}(k,m,1)+b_1^{(3)}(k,m,1)\big]\\
&\qquad+J_2\big[\mathrm{e}^{\mathrm{i}k/3}b_1^{(1)}(k,m-1,1)+\mathrm{e}^{-\mathrm{i}k/3}b_1^{(1)}(k,m+1,1)+\mathrm{e}^{-\mathrm{i}k/3}b_1^{(1)}(k,m-1,2)\\
&\qquad+\mathrm{e}^{\mathrm{i}k/3}b_1^{(1)}(k,m,2)\big]=Eb_1^{(1)}(k,m,1),\\
&\left(6J_1+J_2+\tfrac{3}{2}J_3\right)b_1^{(2)}(k,m,1)+J_4b_1^{(3)}(k,m,1)+J_2\big[\mathrm{e}^{\mathrm{i}k/3}b_1^{(2)}(k,m-1,1)\\
&\qquad+\mathrm{e}^{-\mathrm{i}k/3}b_1^{(2)}(k,m+1,1)+\mathrm{e}^{-\mathrm{i}k/3}b_1^{(2)}(k,m-1,2)+\mathrm{e}^{\mathrm{i}k/3}b_1^{(2)}(k,m,2)\big]\\
&\quad=Eb_1^{(2)}(k,m,1),\\
&\left(6J_1+J_2+\tfrac{3}{2}J_3\right)b_1^{(3)}(k,m,1)+J_4b_1^{(2)}(k,m,1)+J_2\big[\mathrm{e}^{\mathrm{i}k/3}b_1^{(3)}(k,m-1,1)\\
&\qquad+\mathrm{e}^{-\mathrm{i}k/3}b_1^{(3)}(k,m+1,1)+\mathrm{e}^{-\mathrm{i}k/3}b_1^{(3)}(k,m-1,2)+\mathrm{e}^{\mathrm{i}k/3}b_1^{(3)}(k,m,2)\big]\\
&\quad=Eb_1^{(3)}(k,m,1).
\end{aligned}
\tag{91}
$$

Introducing again the unphysical values $b_1^{(j)}(k,m,0)$ and $b_1^{(j)}(k,0,n)$, one can reduce (91) to form (65) by producing the following system of Bethe conditions:

$$
\begin{aligned}
&(\Delta_2+\Delta_1)b_1^{(1)}(k,1,n)+(\Delta_2-\Delta_1)b_1^{(2)}(k,1,n)=\mathrm{e}^{\mathrm{i}k/3}b_1^{(1)}(k,0,n)+\mathrm{e}^{-\mathrm{i}k/3}b_1^{(1)}(k,0,n+1),\\
&(\Delta_2+\Delta_1)b_1^{(2)}(k,1,n)+(\Delta_2-\Delta_1)b_1^{(1)}(k,1,n)\\
&\quad=\mathrm{e}^{\mathrm{i}k/3}b_1^{(2)}(k,0,n)+\mathrm{e}^{-\mathrm{i}k/3}b_1^{(2)}(k,0,n+1),\\
&2\Delta_0b_1^{(3)}(k,1,n)+\tfrac{2}{3}(\Delta_0-\Delta_2)\big[b_1^{(1)}(k,1,n)+b_1^{(2)}(k,1,n)\big]\\
&\quad=\mathrm{e}^{\mathrm{i}k/3}b_1^{(3)}(k,0,n)+\mathrm{e}^{-\mathrm{i}k/3}b_1^{(3)}(k,0,n+1),\\
&2\Delta_0b_1^{(1)}(k,m,1)+\tfrac{2}{3}(\Delta_0-\Delta_2)\big[b_1^{(2)}(k,m,1)+b_1^{(3)}(k,m,1)\big]\\
&\quad=\mathrm{e}^{-\mathrm{i}k/3}b_1^{(1)}(k,m,0)+\mathrm{e}^{\mathrm{i}k/3}b_1^{(1)}(k,m+1,0),\\
&(\Delta_2+\Delta_1)b_1^{(2)}(k,m,1)+(\Delta_2-\Delta_1)b_1^{(3)}(k,m,1)\\
&\quad=\mathrm{e}^{-\mathrm{i}k/3}b_1^{(2)}(k,m,0)+\mathrm{e}^{\mathrm{i}k/3}b_1^{(2)}(k,m+1,0),\\
&(\Delta_2+\Delta_1)b_1^{(3)}(k,m,1)+(\Delta_2-\Delta_1)b_1^{(2)}(k,m,1)\\
&\quad=\mathrm{e}^{-\mathrm{i}k/3}b_1^{(3)}(k,m,0)+\mathrm{e}^{\mathrm{i}k/3}b_1^{(3)}(k,m+1,0).
\end{aligned}
\tag{92}
$$

As in the $S=0$ case, system (92) (as well as the three separate subsystems of form (65) for $b_1^{(j)}(k,m,n)$) is symmetric under a duality transformation

$$
\mathcal{D}\big(b_1^{(j)}(k,m,n)\big)=\bar{b}_1^{(4-j)}(k,n,m).
\tag{93}
$$

Before developing a general analysis of system (92), we shall at once find all the cases when it may be reduced to the $XXZ$-type form (67).

First of all for

$$\Delta_0 = \Delta_1 = \Delta_2, \tag{94}$$

system (92) decouples into three $XXZ$-type subsystems (67) and therefore is completely solvable. So in this case

$$b_1^{(j)}(k, \mathbf{q}, m, n) = \alpha_j b_0(k, \mathbf{q}, m, n), \qquad j = 1, 2, 3, \tag{95}$$

where $\alpha_{1,2,3}$ is a triple of arbitrary parameters. Labeling the coupling constants related to (94) by an upper index '(0)', one may readily obtain from (32)

$$J_d^{(0)} = -J_l^{(0)}, \qquad J_{ll}^{(0)} = 4J_l^{(0)}, \qquad \Delta_{0,1,2}^{(0)} = \frac{J_{rr}^{(0)}}{4J_l^{(0)}} \tag{96}$$

or equivalently

$$J_4^{(0)} = J_5^{(0)} = 0, \qquad \Delta_{0,1,2}^{(0)} = 1 + \frac{3J_3^{(0)}}{2J_2^{(0)}}. \tag{97}$$

In this case according to (17) and (97), an interaction between excited triplet rungs is spin independent. The relation between this model and $XXZ$ chain was studied in more detail in [29].

Besides the complete separable case (94), there are also two configurations of $\Delta$'s for which system (92) possesses a partial solution of form (95) however with special values of the ratios $\alpha_i/\alpha_j$ $(i, j = 1, 2, 3)$. Indeed substituting ansatz (95) into system (92), one readily makes sure that the latter may be reduced to (67) with an appropriate parameter $\Delta$ only under the following system of conditions:

$$\begin{aligned} &(\Delta_1 + \Delta_2 - 2\Delta)\alpha_1 + (\Delta_2 - \Delta_1)\alpha_2 = 0, \\ &(\Delta_2 - \Delta_1)\alpha_1 + (\Delta_1 + \Delta_2 - 2\Delta)\alpha_2 = 0, \end{aligned} \tag{98}$$

$$\begin{aligned} &(\Delta_1 + \Delta_2 - 2\Delta)\alpha_2 + (\Delta_2 - \Delta_1)\alpha_3 = 0, \\ &(\Delta_2 - \Delta_1)\alpha_2 + (\Delta_1 + \Delta_2 - 2\Delta)\alpha_3 = 0, \end{aligned} \tag{99}$$

$$\begin{aligned} &3(\Delta_0 - \Delta)\alpha_1 + (\Delta_0 - \Delta_2)(\alpha_2 + \alpha_3) = 0, \\ &3(\Delta_0 - \Delta)\alpha_3 + (\Delta_0 - \Delta_2)(\alpha_1 + \alpha_2) = 0. \end{aligned} \tag{100}$$

It may be readily observed that the trivial solution of subsystem (98) may be nontrivially extended as a solution of the whole system (98)–(100) only in the case (94). On the other hand, a nontrivial solution of (98), namely $\alpha_1 = \alpha_2$, exists only for $\Delta = \Delta_2$. Moreover for $\Delta_1 = \Delta_2 = \Delta$, system (98) is satisfied for all $\alpha_{1,2}$. Extension of these two solutions to subsystems (99) and (100) results in

$$\alpha_1 = \alpha_2 = \alpha_3, \qquad \Delta = \Delta_0 = \Delta_2, \tag{101}$$

$$4\alpha_1 = -\alpha_2 = 4\alpha_3, \qquad \Delta = \Delta_1 = \Delta_2. \tag{102}$$

Turning to the general ($XXZ$-irreducible) case, we suggest the Bethe ansatz,

$$\begin{aligned} b_1^{(j)}(k, m, n) = {} & B_1^{(j)}(k, \mathbf{q})\,\mathrm{e}^{\mathrm{i}(\tilde{q}_1 m + \tilde{q}_2 n)} - B_2^{(j)}(k, \mathbf{q})\,\mathrm{e}^{\mathrm{i}(\tilde{q}_1 m + (\tilde{q}_1 - \tilde{q}_2) n)} \\ & + B_3^{(j)}(k, \mathbf{q})\,\mathrm{e}^{\mathrm{i}(-\tilde{q}_2 m + (\tilde{q}_1 - \tilde{q}_2) n)} - B_4^{(j)}(k, \mathbf{q})\,\mathrm{e}^{-\mathrm{i}(\tilde{q}_2 m + \tilde{q}_1 n)} \\ & + B_5^{(j)}(k, \mathbf{q})\,\mathrm{e}^{\mathrm{i}((\tilde{q}_2 - \tilde{q}_1) m - \tilde{q}_1 n)} - B_6^{(j)}(k, \mathbf{q})\,\mathrm{e}^{\mathrm{i}((\tilde{q}_2 - \tilde{q}_1) m + \tilde{q}_2 n)}. \end{aligned} \tag{103}$$

The classification of states with complex $\tilde{q}_{1,2}$ has form (82). However each $\mathcal{J}_i$ in (82) is now a set of $l$'s for which all $B_l^{(j)}(k, \mathbf{q}) = 0$. In this paper, we shall not study $S = 1$ and

$S=2$ three-magnon Bethe states with complex wave numbers. For the pure scattering states, the duality (93) reduces on the amplitudes as follows:

$$\mathcal{D}\big(B_l^{(j)}(k,\mathbf{q})\big) = -\bar{B}_{l-3}^{(j)}(k,\mathbf{q}). \tag{104}$$

Substitution of (103) into (92) gives

$$\sum_{l=1}^{18} M_{il}^{(1)}(k,\mathbf{q})B_l(k,\mathbf{q}) = 0, \tag{105}$$

where the vector column $B_l(k,\mathbf{q})$ for $l=1,\ldots,18$ is defined as the following:

$$B_{6(j-1)+m}(k,\mathbf{q}) = B_m^{(j)}(k,\mathbf{q}), \qquad j=1,2,3, \qquad m=1,\ldots,6, \tag{106}$$

while nonzero entries of the $18\times 18$ matrix $M^{(1)}(k,\mathbf{q})$ are as follows:

$$M_{11}^{(1)}(k,\mathbf{q}) = -M_{16,17}^{(1)}(k,\mathbf{q}) = Z(k,\omega_5(\mathbf{q}),\Delta_0),$$

$$M_{22}^{(1)}(k,\mathbf{q}) = M_{88}^{(1)}(k,\mathbf{q}) = -M_{11,12}^{(1)}(k,\mathbf{q}) = -M_{17,18}^{(1)}(k,\mathbf{q}) = Z\left(k,\mathbf{q},\frac{\Delta_1+\Delta_2}{2}\right),$$

$$M_{33}^{(1)}(k,\mathbf{q}) = -M_{18,13}^{(1)}(k,\mathbf{q}) = Z(k,\omega_1(\mathbf{q}),\Delta_0),$$

$$M_{44}^{(1)}(k,\mathbf{q}) = M_{10,10}^{(1)}(k,\mathbf{q}) = -M_{78}^{(1)}(k,\mathbf{q}) = -M_{13,14}^{(1)}(k,\mathbf{q}) = Z\left(k,\omega_2(\mathbf{q}),\frac{\Delta_1+\Delta_2}{2}\right),$$

$$M_{55}^{(1)}(k,\mathbf{q}) = -M_{14,15}^{(1)}(k,\mathbf{q}) = Z(k,\omega_3(\mathbf{q}),\Delta_0),$$

$$M_{66}^{(1)}(k,\mathbf{q}) = M_{12,12}^{(1)}(k,\mathbf{q}) = -M_{9,10}^{(1)}(k,\mathbf{q}) = -M_{15,16}^{(1)}(k,\mathbf{q}) = Z\left(k,\omega_4(\mathbf{q}),\frac{\Delta_1+\Delta_2}{2}\right),$$

$$M_{77}^{(1)}(k,\mathbf{q}) = M_{13,13}^{(1)}(k,\mathbf{q}) = -M_{45}^{(1)}(k,\mathbf{q}) = -M_{10,11}^{(1)}(k,\mathbf{q}) = Z\left(k,\omega_5(\mathbf{q}),\frac{\Delta_1+\Delta_2}{2}\right),$$

$$M_{99}^{(1)}(k,\mathbf{q}) = M_{15,15}^{(1)}(k,\mathbf{q}) = -M_{61}^{(1)}(k,\mathbf{q}) = -M_{12,7}^{(1)}(k,\mathbf{q}) = Z\left(k,\omega_1(\mathbf{q}),\frac{\Delta_1+\Delta_2}{2}\right),$$

$$M_{11,11}^{(1)}(k,\mathbf{q}) = M_{17,17}^{(1)}(k,\mathbf{q}) = -M_{23}^{(1)}(k,\mathbf{q}) = -M_{89}^{(1)}(k,\mathbf{q}) = Z\left(k,\omega_3(\mathbf{q}),\frac{\Delta_1+\Delta_2}{2}\right),$$

$$M_{14,14}^{(1)}(k,\mathbf{q}) = -M_{56}^{(1)}(k,\mathbf{q}) = Z(\mathbf{q},\Delta_0),$$

$$M_{16,16}^{(1)}(k,\mathbf{q}) = -M_{12}^{(1)}(k,\mathbf{q}) = Z(k,\omega_2(\mathbf{q}),\Delta_0),$$

$$M_{18,18}^{(1)}(k,\mathbf{q}) = -M_{34}^{(1)}(k,\mathbf{q}) = Z(k,\omega_4(\mathbf{q}),\Delta_0),$$

$$M_{17}^{(1)}(k,\mathbf{q}) = M_{1,13}^{(1)}(k,\mathbf{q}) = -M_{16,5}^{(1)}(k,\mathbf{q}) = -M_{16,11}^{(1)}(k,\mathbf{q}) = \frac{\Delta_2-\Delta_0}{3}\,\mathrm{e}^{\mathrm{i}q_2},$$

$$M_{18}^{(1)}(k,\mathbf{q}) = M_{1,14}^{(1)}(k,\mathbf{q}) = -M_{16,4}^{(1)}(k,\mathbf{q}) = -M_{16,10}^{(1)}(k,\mathbf{q}) = \frac{\Delta_0-\Delta_2}{3}\,\mathrm{e}^{-\mathrm{i}q_2},$$

$$M_{39}^{(1)}(k,\mathbf{q}) = M_{3,15}^{(1)}(k,\mathbf{q}) = -M_{18,1}^{(1)}(k,\mathbf{q}) = -M_{18,7}^{(1)}(k,\mathbf{q}) = \frac{\Delta_2-\Delta_0}{3}\,\mathrm{e}^{\mathrm{i}q_1},$$

$$M_{3,10}^{(1)}(k,\mathbf{q}) = M_{3,16}^{(1)}(k,\mathbf{q}) = -M_{18,6}^{(1)}(k,\mathbf{q}) = -M_{18,12}^{(1)}(k,\mathbf{q}) = \frac{\Delta_0-\Delta_2}{3}\,\mathrm{e}^{-\mathrm{i}q_1},$$

$$M_{5,11}^{(1)}(k,\mathbf{q}) = M_{5,17}^{(1)}(k,\mathbf{q}) = -M_{14,3}^{(1)}(k,\mathbf{q}) = -M_{14,9}^{(1)}(k,\mathbf{q}) = \frac{\Delta_2-\Delta_0}{3}\,\mathrm{e}^{-\mathrm{i}(q_1+q_2)},$$

$$M_{5,12}^{(1)}(k,\mathbf{q}) = M_{5,18}^{(1)}(k,\mathbf{q}) = -M_{14,2}^{(1)}(k,\mathbf{q}) = -M_{14,8}^{(1)}(k,\mathbf{q}) = \frac{\Delta_0-\Delta_2}{3}\,\mathrm{e}^{\mathrm{i}(q_1+q_2)},$$

$$M_{28}^{(1)}(k,\mathbf{q}) = M_{82}^{(1)}(k,\mathbf{q}) = -M_{11,18}^{(1)}(k,\mathbf{q}) = -M_{17,12}^{(1)}(k,\mathbf{q}) = \frac{\Delta_1-\Delta_2}{2}\,\mathrm{e}^{\mathrm{i}(q_1+q_2)},$$

$$M^{(1)}_{29}(k,\mathbf{q}) = M^{(1)}_{83}(k,\mathbf{q}) = -M^{(1)}_{11,17}(k,\mathbf{q}) = -M^{(1)}_{17,11}(k,\mathbf{q}) = \frac{\Delta_2-\Delta_1}{2}\,\mathrm{e}^{-\mathrm{i}(q_1+q_2)},$$
$$M^{(1)}_{4,10}(k,\mathbf{q}) = M^{(1)}_{10,4}(k,\mathbf{q}) = -M^{(1)}_{7,14}(k,\mathbf{q}) = -M^{(1)}_{13,8}(k,\mathbf{q}) = \frac{\Delta_1-\Delta_2}{2}\,\mathrm{e}^{-\mathrm{i}q_2},$$
$$M^{(1)}_{4,11}(k,\mathbf{q}) = M^{(1)}_{10,5}(k,\mathbf{q}) = -M^{(1)}_{7,13}(k,\mathbf{q}) = -M^{(1)}_{13,7}(k,\mathbf{q}) = \frac{\Delta_2-\Delta_1}{2}\,\mathrm{e}^{\mathrm{i}q_2},$$
$$M^{(1)}_{67}(k,\mathbf{q}) = M^{(1)}_{12,1}(k,\mathbf{q}) = -M^{(1)}_{9,15}(k,\mathbf{q}) = -M^{(1)}_{15,9}(k,\mathbf{q}) = \frac{\Delta_2-\Delta_1}{2}\,\mathrm{e}^{\mathrm{i}q_1},$$
$$M^{(1)}_{6,12}(k,\mathbf{q}) = M^{(1)}_{12,6}(k,\mathbf{q}) = -M^{(1)}_{9,16}(k,\mathbf{q}) = -M^{(1)}_{15,10}(k,\mathbf{q}) = \frac{\Delta_1-\Delta_2}{2}\,\mathrm{e}^{-\mathrm{i}q_1}. \tag{107}$$

As in the case (94), complete solvability of the $S = 1$ problem implies an existence of three independent solutions of system (92). In the Bethe ansatz framework (103), (105), this results in

$$\mathrm{rank}(M^{(1)}(k,\mathbf{q})) = 15 \tag{108}$$

and therefore in

$$P^{(1)}_n(k,\mathbf{q}) = 0, \qquad n = 0, 1, 2. \tag{109}$$

Here, $P^{(1)}_n(k,\mathbf{q})$ are coefficients of the characteristic polynomial:

$$|M^{(1)}(k,\mathbf{q}) - \lambda I| = \sum_{n=0}^{18} P^{(1)}_n(k,\mathbf{q})\lambda^n. \tag{110}$$

Direct calculation based on the computer algebra system MAPLE gives

$$P^{(1)}_0(k,\mathbf{q}) = \det M^{(1)}(k,\mathbf{q}) = 0. \tag{111}$$

So even in the general case $\mathrm{rank}(M^{(1)}(k,\mathbf{q})) \leqslant 17$. As a result system (105) *always* has at at least one solution. Its general form is represented in appendix A.

For the next coefficient $P^{(1)}_1(k,\mathbf{q})$, we have obtained by machinery calculations the following factorization:

$$P^{(1)}_1(k,\mathbf{q}) = \frac{2}{729}(1-\mathrm{e}^{2\mathrm{i}q_1})^2(1-\mathrm{e}^{2\mathrm{i}q_2})^2(1-\mathrm{e}^{2\mathrm{i}(q_1+q_2)})^2\tilde{P}^{(1)}_1(k,\mathbf{q}), \tag{112}$$

where

$$\tilde{P}^{(1)}_1(k,\mathbf{q}) = \mathrm{e}^{-\mathrm{i}(11k+31q_1+31q_2)/3}\sum_{m,n,p\geqslant 0} Q_{m,n,p}(\Delta_0,\Delta_1,\Delta_2)\,\mathrm{e}^{\mathrm{i}(mk+nq_1+pq_2)/3}. \tag{113}$$

The sum in (113) contains 95 052 terms (that is why $\tilde{P}^{(1)}_1(k,\mathbf{q})$ cannot be represented in the format of this paper). According to (112) and (75), condition (109) is satisfied at $n = 1$ either if any two wave numbers in the triple $(k_1, k_2, k_3)$ coincide or if

$$\tilde{P}^{(1)}_1(k,\mathbf{q}) = 0. \tag{114}$$

The former three cases are similar to the case $k_1 = k_2$ in the two-magnon problem studied in section 3. Three-magnon solutions of this type will not be studied in this paper. Turning to equation (114), we shall confine ourselves to the problem of its solvability for all wave numbers. Namely, we shall postulate

$$Q_{m,n,p}(\Delta_0,\Delta_1,\Delta_2) = 0 \tag{115}$$

to be valid at all $m, n, p$.

Despite the fact that system (115) depends only on a triple of unknown variables, it is practically unsolvable by the MAPLE Gröbner package on a personal computer with RAM about 2 Gb. Luckily (as it may be directly checked by machinery calculation)

$$\frac{2Q_{8,4,13} - 9Q_{12,0,11} - Q_{4,8,15}}{2592\Delta_1^2\Delta_2^2\Delta_3^2} = 12(\Delta_0 - \Delta_1)^2 + 15(\Delta_1 - \Delta_2)^2 + 20(\Delta_2 - \Delta_0)^2, \qquad (116)$$

so except (94) there are no solutions with $\Delta_0\Delta_1\Delta_2 \neq 0$.

In each of the three cases $\Delta_{0,1,2} = 0$, the reduced system (115) is essentially simpler than the initial one and may be readily solved on the personal computer. Calculations based on the Gröbner package gave four pairs of solutions. We shall represent them as sets of $\Delta$-parameters: $\mathbf{\Delta} = [\Delta_0, \Delta_1, \Delta_2]$ and additionally as the corresponding sets of the coupling constants: $\mathbf{J} = [J_l, J_d, J_{rr}, J_{ll}, J_{dd}]$ and $\tilde{\mathbf{J}} = [J_2, J_3, J_4, J_5]$. Note that the parameters $J_r$ and $J_l$ remain indefinite. This is rather evident because both of them correspond to the term proportional to $\hat{Q}$ in the Hamiltonian. But according to (23), the former has no affect on the Bethe equations. Namely, the solutions are as follows:

$$\mathbf{\Delta}^{(1,\pm)} = [\pm1, 0, \pm1], \qquad \mathbf{J}^{(1,+)} = [1, 0, 0, 4, 0], \qquad \tilde{\mathbf{J}}^{(1,\pm)} = [\pm1, 0, 1, 1], \qquad (117)$$

$$\mathbf{\Delta}^{(2,\pm)} = [0, \pm1, 0], \qquad \mathbf{J}^{(2,+)} = [1, 0, -2, 4, 0], \qquad \tilde{\mathbf{J}}^{(2,\pm)} = [\pm1, -2, 1, 1], \qquad (118)$$

$$\mathbf{\Delta}^{(3,\pm)} = \left[0, \pm\tfrac{3}{2}, \pm\tfrac{3}{2}\right], \qquad \mathbf{J}^{(3,+)} = [1, 0, 4, 0, -4], \qquad \tilde{\mathbf{J}}^{(3,\pm)} = [\pm1, 4, 0, -1], \qquad (119)$$

$$\mathbf{\Delta}^{(4,\pm)} = \left[\mp\tfrac{3}{2}, 0, 0\right], \qquad \mathbf{J}^{(4,+)} = [1, 0, 1, 0, -4], \qquad \tilde{\mathbf{J}}^{(4,\pm)} = [\pm1, 1, 0, -1]. \qquad (120)$$

It may be readily shown that condition (27) is satisfied only for '+'-type solutions, while the '−' ones may be obtained from them by the symmetry (28). That is why we have omitted representations for $\mathbf{J}^{(1,2,3,4,-)}$. However, they may be readily obtained from $\tilde{\mathbf{J}}^{(1,2,3,4,-)}$ using equations (20).

The models related to $\mathbf{\Delta}^{1,+}$ and $\mathbf{\Delta}^{2,+}$ were first presented in [30]. The former one was intensively studied in [3]. Algebraic structures related to $\mathbf{\Delta}^{1,+}$ and $\mathbf{\Delta}^{3,-}$ models as well as to model (94) with $\Delta_{0,1,2} = 1$ were presented in [23]. (However, the cases (118), (120) and the general case (94) were not discussed in [23].)

As it follows from (101) and (102), all the models (117)–(120) have the $XXZ$-type solution (95). A remaining pair of solutions may be chosen in different ways. (In other words, we do not know the best choice of basis in the two-dimensional solution subspace additional to (95).) The bases obtained by machinery calculations within MAPLE are presented in appendix B.

## 6. $S = 2$ three-magnon sector

A $S = 2$ three-magnon state related to $\mathbf{S}^z = 2$ has the following form:

$$\begin{aligned}|3, 2, k\rangle^2 = \sum_{m<n<p} & e^{ik(m+n+p)/3} \\ & \times \left[b_2^{(1)}(k, n-m, p-n)\cdots|1\rangle_m^+\cdots\left(|1\rangle_n^+\cdots|1\rangle_p^3 - |1\rangle_n^3\cdots|1\rangle_p^+\right)\cdots\right. \\ & \left.+ b_2^{(2)}(k, n-m, p-n)\cdots\left(|1\rangle_m^+\cdots|1\rangle_n^3 - |1\rangle_m^3\cdots|1\rangle_n^+\right)\cdots|1\rangle_p^+\cdots\right. \end{aligned} \qquad (121)$$

For $m, n > 1$, the amplitudes $b_2^{(1,2)}(k, m, n)$ separately satisfy the Schrödinger equation (65) while for $m = 1$ or $n = 1$

$$
\begin{aligned}
&(4J_1+\varepsilon_2)b_2^{(1)}(k,1,n)+J_2\big[\mathrm{e}^{-\mathrm{i}k/3}b_2^{(1)}(k,1,n-1)+\mathrm{e}^{\mathrm{i}k/3}b_2^{(1)}(k,1,n+1)\\
&\qquad+\mathrm{e}^{\mathrm{i}k/3}b_2^{(1)}(k,2,n-1)+\mathrm{e}^{-\mathrm{i}k/3}b_2^{(1)}(k,2,n)\big]=Eb_2^{(1)}(k,1,n),\\
&(4J_1+\varepsilon_1)b_2^{(2)}(k,1,n)+J_4b_2^{(1)}(k,1,n)+J_2\big[\mathrm{e}^{-\mathrm{i}k/3}b_2^{(2)}(k,1,n-1)\\
&\qquad+\mathrm{e}^{\mathrm{i}k/3}b_2^{(2)}(k,1,n+1)+\mathrm{e}^{\mathrm{i}k/3}b_2^{(2)}(k,2,n-1)+\mathrm{e}^{-\mathrm{i}k/3}b_2^{(2)}(k,2,n)\big]\\
&\quad=Eb_2^{(2)}(k,1,n),\\
&(4J_1+\varepsilon_2)b_2^{(2)}(k,m,1)+J_2\big[\mathrm{e}^{\mathrm{i}k/3}b_2^{(2)}(k,m-1,1)+\mathrm{e}^{-\mathrm{i}k/3}b_2^{(2)}(k,m+1,1)\\
&\qquad+\mathrm{e}^{-\mathrm{i}k/3}b_2^{(2)}(k,m-1,2)+\mathrm{e}^{\mathrm{i}k/3}b_2^{(2)}(k,m,2)\big]\\
&\quad=Eb_2^{(2)}(k,m,1),\\
&(4J_1+\varepsilon_1)b_2^{(1)}(k,m,1)+J_4b_2^{(2)}(k,m,1)+J_2\big[\mathrm{e}^{\mathrm{i}k/3}b_2^{(1)}(k,m-1,1)\\
&\qquad+\mathrm{e}^{-\mathrm{i}k/3}b_2^{(1)}(k,m+1,1)+\mathrm{e}^{-\mathrm{i}k/3}b_2^{(1)}(k,m-1,2)+\mathrm{e}^{\mathrm{i}k/3}b_2^{(1)}(k,m,2)\big]\\
&\quad=Eb_2^{(1)}(k,m,1).
\end{aligned}
\tag{122}
$$

Introducing again the unphysical amplitudes, we obtain from (122) the corresponding system of coupled Bethe conditions

$$
\begin{aligned}
&2\Delta_1b_2^{(1)}(k,m,1)+(\Delta_2-\Delta_1)b_2^{(2)}(k,m,1)=\mathrm{e}^{-\mathrm{i}k/3}b_2^{(1)}(k,m,0)+\mathrm{e}^{\mathrm{i}k/3}b_2^{(1)}(k,m+1,0),\\
&2\Delta_2b_2^{(2)}(k,m,1)=\mathrm{e}^{-\mathrm{i}k/3}b_2^{(2)}(k,m,0)+\mathrm{e}^{\mathrm{i}k/3}b_2^{(2)}(k,m+1,0),\\
&2\Delta_1b_2^{(2)}(k,1,n)+(\Delta_2-\Delta_1)b_2^{(1)}(k,1,n)=\mathrm{e}^{\mathrm{i}k/3}b_2^{(2)}(k,0,n)+\mathrm{e}^{-\mathrm{i}k/3}b_2^{(2)}(k,0,n+1),\\
&2\Delta_2b_2^{(1)}(k,1,n)=\mathrm{e}^{\mathrm{i}k/3}b_2^{(1)}(k,0,n)+\mathrm{e}^{-\mathrm{i}k/3}b_2^{(1)}(k,0,n+1),
\end{aligned}
\tag{123}
$$

invariant under duality transformation

$$\mathcal{D}\big(b_2^{(j)}(k,m,n)\big)=\bar{b}_2^{(3-j)}(k,n,m). \tag{124}$$

For

$$\Delta_1=\Delta_2, \tag{125}$$

or (according to (32))

$$J_{ll}=-4J_d, \tag{126}$$

this system decouples into a pair of the $XXZ$-type subsystems (67) on $b_2^{(1,2)}(k,m,n)$. In this case, the general solution

$$b_2^{(j)}(k,\mathbf{q},m,n)=\beta_jb_0(k,\mathbf{q},m,n),\qquad j=1,2, \tag{127}$$

depends on $\mathbf{q}$ and two arbitrary parameters $\beta_{1,2}$.

It may be readily proved that the $XXZ$-type solutions (127) exist only under condition (125).

In the general case making the standard substitution

$$
\begin{aligned}
b_2^{(j)}(k,m,n)&=C_1^{(j)}(k,\mathbf{q})\,\mathrm{e}^{\mathrm{i}(\tilde{q}_1m+\tilde{q}_2n)}-C_2^{(j)}(k,\mathbf{q})\,\mathrm{e}^{\mathrm{i}(\tilde{q}_1m+(\tilde{q}_1-\tilde{q}_2)n)}\\
&\quad+C_3^{(j)}(k,\mathbf{q})\,\mathrm{e}^{\mathrm{i}(-\tilde{q}_2m+(\tilde{q}_1-\tilde{q}_2)n)}-C_4^{(j)}(k,\mathbf{q})\,\mathrm{e}^{-\mathrm{i}(\tilde{q}_2m+\tilde{q}_1n)}\\
&\quad+C_5^{(j)}(k,\mathbf{q})\,\mathrm{e}^{\mathrm{i}((\tilde{q}_2-\tilde{q}_1)m-\tilde{q}_1n)}-C_6^{(j)}(k,\mathbf{q})\,\mathrm{e}^{\mathrm{i}((\tilde{q}_2-\tilde{q}_1)m+\tilde{q}_2n)},
\end{aligned}
\tag{128}
$$

one results in a linear system

$$\sum_{j=1}^{12}M_{ij}^{(2)}(k,\mathbf{q})C_j(k,\mathbf{q})=0, \tag{129}$$

where in a manner similar to equation (106)

$$C_{6(j-1)+m}(k,\mathbf{q}) = C_m^{(j)}(k,\mathbf{q}), \qquad j=1,2, \quad m=1,\ldots,6, \tag{130}$$

and the $12\times 12$ matrix $M^{(2)}(k,\mathbf{q})$ has the following nonzero entries:

$$\begin{aligned}
&M_{11}^{(2)}(k,\mathbf{q}) = -M_{10,11}^{(2)}(k,\mathbf{q}) = Z(k,\omega_5(\mathbf{q}),\Delta_1),\\
&M_{22}^{(2)}(k,\mathbf{q}) = -M_{11,12}^{(2)}(k,\mathbf{q}) = Z(k,\mathbf{q},\Delta_2),\\
&M_{33}^{(2)}(k,\mathbf{q}) = -M_{12,7}^{(2)}(k,\mathbf{q}) = Z(k,\omega_1(\mathbf{q}),\Delta_1),\\
&M_{44}^{(2)}(k,\mathbf{q}) = -M_{78}^{(2)}(k,\mathbf{q}) = Z(k,\omega_2(\mathbf{q}),\Delta_2),\\
&M_{55}^{(2)}(k,\mathbf{q}) = -M_{89}^{(2)}(k,\mathbf{q}) = Z(k,\omega_3(\mathbf{q}),\Delta_1),\\
&M_{66}^{(2)}(k,\mathbf{q}) = -M_{9,10}^{(2)}(k,\mathbf{q}) = Z(k,\omega_4(\mathbf{q}),\Delta_2),\\
&M_{77}^{(2)}(k,\mathbf{q}) = -M_{45}^{(2)}(k,\mathbf{q}) = Z(k,\omega_5(\mathbf{q}),\Delta_2),\\
&M_{88}^{(2)}(k,\mathbf{q}) = -M_{56}^{(2)}(k,\mathbf{q}) = Z(k,\mathbf{q},\Delta_1),\\
&M_{99}^{(2)}(k,\mathbf{q}) = -M_{61}^{(2)}(k,\mathbf{q}) = Z(k,\omega_1(\mathbf{q}),\Delta_2),\\
&M_{10,10}^{(2)}(k,\mathbf{q}) = -M_{12}^{(2)}(k,\mathbf{q}) = Z(k,\omega_2(\mathbf{q}),\Delta_1),\\
&M_{11,11}^{(2)}(k,\mathbf{q}) = -M_{23}^{(2)}(k,\mathbf{q}) = Z(k,\omega_3(\mathbf{q}),\Delta_2),\\
&M_{12,12}^{(2)}(k,\mathbf{q}) = -M_{34}^{(2)}(k,\mathbf{q}) = Z(k,\omega_4(\mathbf{q}),\Delta_1),\\
&M_{17}^{(2)}(k,\mathbf{q}) = -M_{10,5}^{(2)}(k,\mathbf{q}) = \frac{\Delta_1-\Delta_2}{2}\,\mathrm{e}^{\mathrm{i}q_2},\\
&M_{18}^{(2)}(k,\mathbf{q}) = -M_{10,4}^{(2)}(k,\mathbf{q}) = \frac{\Delta_2-\Delta_1}{2}\,\mathrm{e}^{-\mathrm{i}q_2},\\
&M_{39}^{(2)}(k,\mathbf{q}) = -M_{12,1}^{(2)}(k,\mathbf{q}) = \frac{\Delta_1-\Delta_2}{2}\,\mathrm{e}^{\mathrm{i}q_1},\\
&M_{3,10}^{(2)}(k,\mathbf{q}) = -M_{12,6}^{(2)}(k,\mathbf{q}) = \frac{\Delta_2-\Delta_1}{2}\,\mathrm{e}^{-\mathrm{i}q_1},\\
&M_{5,11}^{(2)}(k,\mathbf{q}) = -M_{83}^{(2)}(k,\mathbf{q}) = \frac{\Delta_1-\Delta_2}{2}\,\mathrm{e}^{-\mathrm{i}(q_1+q_2)},\\
&M_{5,12}^{(2)}(k,\mathbf{q}) = -M_{82}^{(2)}(k,\mathbf{q}) = \frac{\Delta_2-\Delta_1}{2}\,\mathrm{e}^{\mathrm{i}(q_1+q_2)}.
\end{aligned} \tag{131}$$

As in the $S=1$ case, we shall deal only with the pure scattering states for which the duality (124) gives

$$\mathcal{D}\big(C_l^{(j)}(k,\mathbf{q})\big) = -\bar{C}_{l-3}^{(j)}(k,\mathbf{q}). \tag{132}$$

To be completely solvable, system (129) must possess two independent solutions. Equivalently, there should be

$$\mathrm{rank}(M^{(2)}(k,\mathbf{q})) = 10. \tag{133}$$

According to machinery calculation

$$\det M^{(2)}(k,\mathbf{q}) = -6(\Delta_1-\Delta_2)^2(1-\mathrm{e}^{2\mathrm{i}(q_1+q_2)})^2(1-\mathrm{e}^{-2\mathrm{i}q_1})^2(1-\mathrm{e}^{-2\mathrm{i}q_2})^2Y^2(k,\mathbf{q}), \tag{134}$$

where

$$Y(k,\mathbf{q}) = [(\Delta_1-\Delta_2)^2-1]\cos k + \frac{[(\Delta_1+\Delta_2)^2-1][E(k,\mathbf{q})-6J_1]}{2J_2} - 4\Delta_1\Delta_2(\Delta_1+\Delta_2), \tag{135}$$

and $E(k,\mathbf{q})$ is determined by (72), (37) and (75).

Condition $\det M^{(2)}(k,\mathbf{q})=0$ will be satisfied at all $k$, $q_1$ and $q_2$, either in the case (125) or in the four additional ones

$$\Delta_1=\pm 1,\qquad \Delta_2=0 \tag{136}$$

and

$$\Delta_1=0,\qquad \Delta_2=\pm 1. \tag{137}$$

Machinery calculations show that in all these cases, condition (133) is satisfied. The corresponding solutions of system (129) are presented in the appendix.

According to (135), there is also a special point on the $(k,E)$ plane:

$$E(k,\mathbf{q})=6J_1+\frac{2J_2}{(\Delta_1+\Delta_2)^2-1}[4\Delta_1\Delta_2(\Delta_1+\Delta_2)+(1-(\Delta_1-\Delta_2)^2)\cos k], \tag{138}$$

in which system (129) is solvable. However, this special solution is not studied in this paper.

## 7. Integrability and the Reshetikhin condition

An alternative to the coordinate Bethe ansatz is the so-called algebraic Bethe ansatz or the inverse scattering method [15–17]. It is based on the representation of the finite dimensional matrix $H$ related to the local Hamiltonian density $H_{n,n+1}$ as a derivative of the corresponding $R$-matrix:

$$H=\frac{\partial}{\partial\lambda}\check{R}(\lambda)\Big|_{\lambda=0}. \tag{139}$$

The latter satisfies the Yang–Baxter equation

$$\check{R}_{12}(\lambda-\mu)\check{R}_{23}(\lambda)\check{R}_{12}(\mu)=\check{R}_{23}(\mu)\check{R}_{12}(\lambda)\check{R}_{23}(\lambda-\mu), \tag{140}$$

and the initial condition

$$\check{R}(0)\propto I \tag{141}$$

(where again $I$ is an identity matrix).

From (139)–(141), the Reshetikhin condition follows [17]:

$$[H_{12}+H_{23},[H_{12},H_{23}]]=K_{23}-K_{12}. \tag{142}$$

For the Hamiltonian density (17) with $J_6=0$, it gives the following system of equations:

$$\begin{aligned}
&J_2J_4(J_1+J_3+J_5)=0,\\
&J_2(J_4-J_5)(J_4+J_5)=0,\\
&J_2(J_4-J_5)(2J_1+2J_3-J_4+5J_5)=0,\\
&(J_4-J_5)\left(J_2^2-J_5^2+2J_4J_5\right)=0,\\
&J_5\left(J_2^2-2J_4^2-J_5^2+2J_4J_5\right)=0,\\
&J_2\left(J_3^2+2J_1J_3-4J_5^2+4J_4J_5\right)=0.
\end{aligned} \tag{143}$$

Taking at the first $J_2=0$, one gets from (143) $J_5=0$. This case with degenerate one-magnon dispersion is of poor physical interest and was already studied in [28]. Now taking $J_2\neq 0$ and using (32), one can subdivide system (143) into two subsystems:

$$\begin{aligned}
&(\Delta_2-\Delta_0)(\Delta_2-\Delta_1)(3\Delta_1-2\Delta_2-\Delta_0)=0,\\
&(\Delta_2-\Delta_0)[9(\Delta_1-\Delta_2)^2-4(\Delta_2-\Delta_0)^2+9]=0,\\
&(3\Delta_1-\Delta_2-2\Delta_0)[9(\Delta_1-\Delta_2)^2+4(\Delta_2-\Delta_0)^2-9]=0
\end{aligned} \tag{144}$$

and

$$\begin{aligned}&(\Delta_1 - \Delta_2)(3\Delta_r + 3\Delta_1 + \Delta_2 - \Delta_0) = 0,\\&(\Delta_2 - \Delta_0)(3\Delta_r + \Delta_2 - 3\Delta_1 + 5\Delta_0) = 0,\\&3\Delta_r(3\Delta_1 + \Delta_2 - \Delta_0) + 9\Delta_1^2 - 18\Delta_1\Delta_2 + \Delta_2^2 + 4\Delta_2\Delta_0 - 5\Delta_0^2 = 0,\end{aligned} \tag{145}$$

where $\Delta_r \equiv J_{\mathrm{r}}/(J_{\mathrm{l}} - J_d)$.

We have separated (145) and (144) because according to (23), $J_{\mathrm{r}}$ (related to the term proportional to $\hat{Q}$) has no real affect on integrability. A similar situation occurs for the $XXZ$ chain in a longitudinal magnetic field $h$. Although the system is integrable, the corresponding $R$-matrix exists only for $h = 0$.

Subsystem (144) has three solutions. The first one is solution (94) for which subsystem (145) is also solvable. The remaining two solutions of (144) are as follows:

$$\Delta_0 = \Delta_2, \qquad (\Delta_1 - \Delta_2)^2 = 1, \tag{146}$$

$$\Delta_1 = \Delta_2, \qquad 4(\Delta_0 - \Delta_2)^2 = 9. \tag{147}$$

A substitution of (146) into (145) shows that the latter subsystem is solvable with respect to $\Delta_r$ only for

$$\Delta_1\Delta_2 = 0. \tag{148}$$

Equations (146) and (148) together result in (117) and (118).

Analogously, a substitution of (147) into (145) gives

$$\Delta_1\Delta_0 = 0. \tag{149}$$

Equations (147) and (149) together result in (119) and (120).

Corresponding to the integrable cases, $R$-matrices were already presented in [21] within the following basis in space $\mathbb{C}^{16}$:

$$\begin{aligned}&f_{3(i-1)+j} = e_i \otimes e_j, \qquad f_{9+i} = |0\rangle \otimes e_j,\\&f_{12+i} = e_i \otimes |0\rangle, \qquad f_{16} = |0\rangle \otimes |0\rangle.\end{aligned} \tag{150}$$

Here $i, j = 1, 2, 3$ and $e_1 = |1\rangle^{+1}$, $e_2 = |1\rangle^0$ and $e_3 = |1\rangle^{-1}$.

In this basis, the $R$-matrix corresponding to (94) has the block $XXZ$-type form:

$$\check{R}^{(0)}(\lambda) = \begin{pmatrix}\sinh(\lambda+\eta)I_9 & 0 & 0 & 0\\ 0 & \sinh\eta I_3 & \sinh\lambda I_3 & 0\\ 0 & \sinh\lambda I_3 & \sinh\eta I_3 & 0\\ 0 & 0 & 0 & \sinh(\lambda+\eta)\end{pmatrix}. \tag{151}$$

For a very special value of $\eta$, it was also presented in [29].

In the cases (117) (for $J_{\mathrm{r}} = 0$) and (118) ($J_{\mathrm{r}} = J_{\mathrm{l}}$), the matrices $H$ are correspondingly normal and graded $\mathbb{C}^4 \otimes \mathbb{C}^4$ permutators $\mathcal{P}_4$ and $\tilde{\mathcal{P}}_4$. (In the latter case, the subspace generated by $|0\rangle$ has negative grading.) The related $R$-matrices have a rather simple form

$$\check{R}^{(1,2)}(\lambda) = \eta I_{16} + \lambda H. \tag{152}$$

Integrability of these models was first noted in [30]. The case (117) was intensively studied in [3].

The $R$-matrices related to (119) (for $J_r = J_l$) and (120) ($2J_r = 5J_l$) also have block forms:

$$\check{R}^{(3)}(\lambda) = \begin{pmatrix} r(\lambda,\eta_0) & 0 & 0 & 0 \\ 0 & \sinh\eta_0 I_3 & \sinh\lambda I_3 & 0 \\ 0 & \sinh\lambda I_3 & \sinh\eta_0 I_3 & 0 \\ 0 & 0 & 0 & \sinh(\lambda+\eta_0) \end{pmatrix},$$
$$\check{R}^{(4)}(\lambda) = \begin{pmatrix} r(\lambda,\eta_0) & 0 & 0 & 0 \\ 0 & \sinh\eta_0 I_3 & \sinh\lambda I_3 & 0 \\ 0 & \sinh\lambda I_3 & \sinh\eta_0 I_3 & 0 \\ 0 & 0 & 0 & \sinh(\eta_0-\lambda) \end{pmatrix}, \tag{153}$$

where $\sinh\eta_0 = \sqrt{5}/2$ and

$$r(\lambda,\eta_0) = \begin{pmatrix} f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & f & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f-g & 0 & g & 0 & -g & 0 & 0 \\ 0 & 0 & 0 & f & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g & 0 & f-g & 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 \\ 0 & 0 & -g & 0 & g & 0 & f-g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f \end{pmatrix} \tag{154}$$

($f = \sinh(\lambda+\eta_0)$, $g = \sinh\lambda$).

The matrix $r(\lambda,\eta_0)$ itself satisfies the Yang–Baxter equation and describes the $S = 1$ biquadratic spin chain. As was shown in [31], this $R$-matrix as well as its generalization (related to arbitrary $\eta$) is related to the Temperley–Lieb algebra.

## 8. Action of the $S_3$ group on the eigenspaces

As will be shown below (see equation (162)), the $S_3$-action (87) in the $q$-space results in corresponding symmetry of Bethe wavefunctions. The latter is useful (see appendix A) for compact representation of amplitudes.

First of all, let us consider the case $S = 0$ (which is analogous to $S = 3$). The matrix $M^{(0)}(k,\mathbf{q})$ possesses the following symmetry:

$$M^{(0)}(k,\omega_j(\mathbf{q})) = J_L^{(0)}(\omega_j) M^{(0)}(k,\mathbf{q}) J_R^{(0)}(\omega_j), \tag{155}$$

where the matrices $J_L^{(0)}(\omega_j)$ and $J_R^{(0)}(\omega_j)$ give left and right representations of the group $S_3$:

$$J_L^{(0)}(\omega_i)J_L^{(0)}(\omega_j) = J_L^{(0)}(\omega_i\cdot\omega_j), \qquad J_R^{(0)}(\omega_i)J_R^{(0)}(\omega_j) = J_R^{(0)}(\omega_j\cdot\omega_i). \tag{156}$$

Explicit expressions for the matrices $J_{L,R}^{(0)}(\omega_j)$ may be obtained from equations (79), (156) and the following representations for generators:

$$J_L^{(0)}(\omega_1) = \begin{pmatrix} 1 & \mathbb{O}_{1,5} \\ \mathbb{O}_{5,1} & \tilde{I}_5 \end{pmatrix}, \qquad J_L^{(0)}(\omega_5) = \begin{pmatrix} \tilde{I}_5 & \mathbb{O}_{5,1} \\ \mathbb{O}_{1,5} & 1 \end{pmatrix},$$
$$J_R^{(0)}(\omega_1) = -\begin{pmatrix} \tilde{I}_2 & \mathbb{O}_{2,4} \\ \mathbb{O}_{4,2} & \tilde{I}_4 \end{pmatrix}, \qquad J_R^{(0)}(\omega_5) = -\tilde{I}_6. \tag{157}$$

Here by $\mathbb{O}_{m,n}$, we denote an $m\times n$ matrix with all zero entries while by $\tilde{I}_n$ an $n\times n$ matrix with units in the second diagonal (and all other entries equal to zero).

Similar relations

$$M^{(1,2)}(k, \omega_j(\mathbf{q})) = J_{\mathrm{L}}^{(1,2)}(\omega_j) M^{(1,2)}(k, \mathbf{q}) J_{\mathrm{R}}^{(1,2)}(\omega_j), \tag{158}$$

with

$$J_{\mathrm{L,R}}^{(1)} = I_3 \otimes J_{\mathrm{L,R}}^{(0)}, \qquad J_{\mathrm{L,R}}^{(2)} = I_2 \otimes J_{\mathrm{L,R}}^{(0)}, \tag{159}$$

are also valid for $M^{(1,2)}(k, \mathbf{q})$ given by (107) and (131).

The symmetry (158) allows us to produce new solutions to equation (105) or (129) from the known one (for equation (84), the result is trivial). Indeed if

$$M^{(n)}(k, \mathbf{q}) F^{(n)}(k, \mathbf{q}) = 0, \tag{160}$$

for some vector $F^{(n)}(k, \mathbf{q})$ $(\dim(F^{(1)}(k, \mathbf{q})) = 18, \dim(F^{(2)}(k, \mathbf{q})) = 12)$, then according to (158)

$$M^{(n)}(k, \mathbf{q}) J_{\mathrm{R}}^{(n)}(\omega_j) F^{(n)}(k, \omega_j(\mathbf{q})) = 0. \tag{161}$$

In other words, we have obtained the following action of group $S_3$ on the eigenspaces:

$$\omega_j(F^{(n)})(k, \mathbf{q}) = J_{\mathrm{R}}^{(n)}(\omega_j) F^{(n)}(k, \omega_j(\mathbf{q})). \tag{162}$$

Here, $\omega_j(F^{(n)})$ is a vector related to a new solution (which in fact may coincide with the present one).

## 9. Summary

In this paper, we analyzed two- and three-magnon problems for a rung-dimerized spin ladder. It was shown that the Bethe form of the two-magnon solution may be obtained in a straightforward manner from the corresponding Schrödinger equation.

The three-magnon problem was first analyzed in general terms in all sectors of total spin $S = 0, 1, 2, 3$. For all $S$, it was shown that the Schrödinger equation reduced to the center-of-mass frame is invariant under an appropriate duality transformation. On the other hand, Fourier substitution (70) directly results in the Bethe form of the wavefunction.

Applicability of the Bethe ansatz for the three-magnon problem was analyzed separately in all sectors of total spin. It was shown that for $S = 0$ and $S = 3$, the problem is always solvable and the corresponding solution has a form typical of the $XXZ$ model. The sector $S = 1$ is completely solvable in the five cases (94) and (117)–(120). Nevertheless, a special partial solution (see appendix A) exists for all values of the coupling constants. The sector $S = 2$ is solvable under one of conditions (125), (136), (137) or (138). Explicit expressions for the solutions related to (136) and (137) are presented in appendix C.

The result was compared with the previous consideration based on the solvability analysis for the Yang–Baxter equation. It was shown that the three-magnon problem for the Hamiltonian $\hat{H}$ is completely solvable within the coordinate Bethe ansatz if and only if the corresponding $R$-matrix exists for some point in the orbit $\hat{H} + \alpha \hat{Q}$ ($\alpha$ is real).

Finally, it was shown that the $S_3$-symmetry of the Bethe ansatz equations results in action (162) of group $S_3$ on the space of Bethe vectors.

## Acknowledgments

The author is very grateful to P P Kulish and M I Vyazovsky for helpful discussions.

## Appendix A. Partial solution in the $S = 1$ sector

An explicit form of the special partial solution of equation (105) obtained by MAPLE is rather complicated. For example, the expressions for $B_j(k,\mathbf{q})$ at $j = 1,\ldots,6$ and $j = 13,\ldots,18$ contain 1106 terms while the expression for $B_j(k,\mathbf{q})$ at $j = 7,\ldots,12$ contains 1090.

Since in a general case this solution is a single one, it must be $S_3$-symmetric and auto- (or anti-auto) dual. It may be readily proved that these symmetry properties allow us to obtain all components from $B_1(k,\mathbf{q})$ and $B_7(k,\mathbf{q})$ using equations (104) and (162). Below, we give representations of these two components for the solution presented in an anti-autodual form.

First of all, $B_1(k,\mathbf{q})$ possesses the following decomposition:

$$B_1(k,\mathbf{q}) = B_1^{(s)}(k,\mathbf{q}) + B_1^{(a)}(k,\mathbf{q}), \tag{A.1}$$

where the term $B_1^{(s)}(k,\mathbf{q})$ is symmetric under the transposition,

$$k \to -k, \qquad q_1 \leftrightarrow q_2, \tag{A.2}$$

while $B_1^{(a)}(k,\mathbf{q})$ is antisymmetric.

For $B_1^{(s)}(k,\mathbf{q})$, we have the following representation:

$$\begin{aligned} B_1^{(s)}(k,\mathbf{q}) = {} & \frac{45}{2}F(\Delta_1,\Delta_2,\Delta_0) + \frac{45}{2}F(\Delta_2,\Delta_0,\Delta_1) + \frac{9}{2}F(\Delta_0,\Delta_1,\Delta_2) \\ & + u_1u_2u_3\left[W_1u_1u_2u_3 + W_2\frac{u_1u_2}{z_1z_2} + W_3\left(\frac{u_1z_2}{z_1^2} + \frac{u_2z_1}{z_2^2}\right)u_3\right. \\ & \left. + W_4\frac{u_3}{z_1z_2} + W_5\left(\frac{u_1}{z_1^3} + \frac{u_2}{z_2^3}\right) + \frac{W_6}{z_1^2z_2^2}\right], \end{aligned} \tag{A.3}$$

where

$$\begin{aligned} F(\Delta,\Delta',\Delta'') = {} & (\Delta' - \Delta)Z(k,\mathbf{q},\Delta'')Z(k,\omega_2(\mathbf{q}),\Delta'')Z(k,\omega_4(\mathbf{q}),\Delta'') \\ & \cdot [u_1u_2u_3 + \Delta\Delta'\cos k + \Delta\Delta'(\Delta + \Delta')]. \end{aligned} \tag{A.4}$$

The parameters

$$\begin{aligned} u_1 &= \cos\left(\frac{k}{3} + \frac{\tilde{q}_1}{2}\right), & u_2 &= \cos\left(\frac{k}{3} - \frac{\tilde{q}_2}{2}\right), \\ u_3 &= \cos\left(\frac{k}{3} + \frac{\tilde{q}_2 - \tilde{q}_1}{2}\right), & z_j &= \mathrm{e}^{\mathrm{i}\tilde{q}_j/2} \end{aligned} \tag{A.5}$$

have a simpler form being expressed from $\tilde{q}_{1,2}$.

The coefficients $W_j$ for $j = 1, 2, 3$ are the following:

$$\begin{aligned} W_1 = {} & 27\Delta_1^3 - 5\Delta_2^3 + 8\Delta_0^3 + 45\Delta_1^2\Delta_2 - 75\Delta_1\Delta_2^2 - 90\Delta_2^2\Delta_0 \\ & + 60\Delta_2\Delta_0^2 - 18\Delta_1^2\Delta_0 - 12\Delta_1\Delta_0^2 + 60\Delta_1\Delta_2\Delta_0, \\ W_2 = {} & 18\Delta_1^3\Delta_2 - 10\Delta_1\Delta_2^3 - 45\Delta_1^3\Delta_0 - 50\Delta_1\Delta_0^3 + 15\Delta_2^3\Delta_0 + 42\Delta_2\Delta_0^3 \\ & + 30\big(3\Delta_1^2 - \Delta_2^2\big)\Delta_0^2 + 3\Delta_1\Delta_2\Delta_0(65\Delta_2 - 39\Delta_1 - 36\Delta_0), \\ W_3 = {} & W_2 + 15(\Delta_1 - \Delta_2)(\Delta_1 - \Delta_0)(\Delta_2 - \Delta_0)(4\Delta_0 - 3\Delta_1 - \Delta_2). \end{aligned} \tag{A.6}$$

For $j = 4, 5, 6$, they may be obtained from (A.6) by the following formulae (observed purely empirically):

$$W_4 = \frac{\varphi(W_2)}{\Delta_1\Delta_2\Delta_0}, \qquad W_5 = \frac{\varphi(W_3)}{\Delta_1\Delta_2\Delta_0}, \qquad W_6 = \varphi(W_1), \tag{A.7}$$

where the homomorphism $\varphi$ is defined as follows:

$$\varphi(\Delta_j) = \frac{\Delta_1\Delta_2\Delta_0}{\Delta_j}. \tag{A.8}$$

For $B_1^{(a)}(k,\mathbf{q})$, we found the following representation:

$$B_1^{(a)}(k,\mathbf{q}) = \tfrac{15}{2}(\Delta_1-\Delta_2)(\Delta_1-\Delta_0)(\Delta_2-\Delta_0)u_3\tilde{B}_1^{(a)}(k,\mathbf{q}), \tag{A.9}$$

where

$$\begin{aligned}\tilde{B}_1^{(a)}(k,\mathbf{q}) = u_1u_2\Bigg[&(3-4\Delta_1\Delta_2-6\Delta_0\Delta_2-2\Delta_0\Delta_1)\left(\frac{u_1}{z_1^3}-\frac{u_2}{z_2^3}\right)\\ &+3\mathrm{i}\sin k+3\mathrm{i}(1+4(\Delta_0\Delta_1+\Delta_1\Delta_2+\Delta_0\Delta_2))\frac{v_3}{z_1z_2}\\ &-2\mathrm{i}(2\Delta_0+3\Delta_1+\Delta_2)\left(\frac{u_1z_2}{z_1^2}+\frac{u_2z_1}{z_2^2}\right)v_3\\ &-2\mathrm{i}(\Delta_0+2\Delta_2)\frac{u_1v_2+u_2v_1}{z_1z_2}\Bigg]-6i\,\Delta_0\Delta_1\Delta_2\frac{u_1v_2+u_2v_1}{z_1^2z_2^2},\end{aligned} \tag{A.10}$$

and

$$v_1=\sin\left(\frac{k}{3}+\frac{\tilde{q}_1}{2}\right),\qquad v_2=\sin\left(\frac{k}{3}-\frac{\tilde{q}_2}{2}\right),\qquad v_3=\sin\left(\frac{k}{3}+\frac{\tilde{q}_2-\tilde{q}_1}{2}\right). \tag{A.11}$$

Representation of $B_7(k,\mathbf{q})$ is similar to (A.3):

$$\begin{aligned}B_7(k,\mathbf{q}) = {}&45F(\Delta_0,\Delta_2,\Delta_1)+27F(\Delta_0,\Delta_1,\Delta_2)\\ &+u_1u_2u_3\Bigg[\tilde{W}_1u_1u_2u_3+\tilde{W}_2\frac{u_1u_2}{z_1z_2}+\tilde{W}_3\left(\frac{u_1z_2}{z_1^2}+\frac{u_2z_1}{z_2^2}\right)u_3\\ &+\tilde{W}_4\frac{u_3}{z_1z_2}+\tilde{W}_5\left(\frac{u_1}{z_1^3}+\frac{u_2}{z_2^3}\right)+\frac{\tilde{W}_6}{z_1^2z_2^2}\Bigg],\end{aligned} \tag{A.12}$$

where

$$\begin{aligned}\tilde{W}_1 &= 27\Delta_1^3+5\Delta_2^3-32\Delta_0^3+45\Delta_1\Delta_2(\Delta_2-\Delta_1)\\ &\quad+72\Delta_1\Delta_0(\Delta_1-\Delta_0)+120\Delta_2\Delta_0(\Delta_2-\Delta_0),\\ \tilde{W}_2 &= 45\Delta_1^3\Delta_0-72\Delta_1^3\Delta_2-80\Delta_2^3\Delta_1+75\Delta_2^3\Delta_0+20\Delta_0^3\Delta_1+12\Delta_0^3\Delta_2\\ &\quad+60\big(2\Delta_1^2\Delta_2^2-\Delta_1^2\Delta_0^2-\Delta_2^2\Delta_0^2\big)+3\Delta_1\Delta_2\Delta_0(104\Delta_0-29\Delta_1-75\Delta_2),\\ \tilde{W}_3 &= W_2+90(\Delta_1-\Delta_2)^2(\Delta_1-\Delta_0)(\Delta_2-\Delta_0).\end{aligned} \tag{A.13}$$

Again, the parameters $\tilde{W}_{4,5,6}$ may be obtained from $\tilde{W}_{1,2,3}$ according to (A.7) and (A.8).

## Appendix B. Additional $S = 1$ solutions

We shall use here the following notations:

$$\begin{aligned}&m_1(k,\mathbf{q},\Delta)=Z(k,\omega_5(\mathbf{q}),\Delta), &&m_2(k,\mathbf{q},\Delta)=Z(k,\mathbf{q},\Delta),\\ &m_3(k,\mathbf{q},\Delta)==Z(k,\omega_1(\mathbf{q}),\Delta), &&m_4(k,\mathbf{q},\Delta)=Z(k,\omega_2(\mathbf{q}),\Delta),\\ &m_5(k,\mathbf{q},\Delta)=Z(k,\omega_3(\mathbf{q}),\Delta), &&m_6(k,\mathbf{q},\Delta)=Z(k,\omega_4(\mathbf{q}),\Delta)\end{aligned} \tag{B.1}$$

(for definition of $Z(k,\mathbf{q},\Delta)$ and $\omega_j(\mathbf{q})$, see (86) and (87)).

For $\Delta_0=\Delta_2=1$, $\Delta_1=0$, the space of solutions additional to (95) and (101) is generated by the vector

$$\begin{aligned}B_1^{(1)}(k,\mathbf{q}) &= 2m_4(k,\mathbf{q},1)m_6(k,\mathbf{q},1)\sin\frac{k+2q_1+4q_2}{6}\sin\frac{k-4q_1-2q_2}{6},\\ B_2^{(1)}(k,\mathbf{q}) &= 2m_1(k,\mathbf{q},1)m_6(k,\mathbf{q},1)\sin\frac{k+2q_1+4q_2}{6}\sin\frac{k-4q_1-2q_2}{6},\end{aligned}$$

$$
\begin{aligned}
&B_3^{(1)}(k,\mathbf{q}) = -m_1(k,\mathbf{q},1)m_2(k,\mathbf{q},1)m_6(k,\mathbf{q},1),\\
&B_4^{(1)}(k,\mathbf{q}) = -m_1(k,\mathbf{q},1)m_2(k,\mathbf{q},1)m_3(k,\mathbf{q},1),\\
&B_5^{(1)}(k,\mathbf{q}) = 2m_2(k,\mathbf{q},1)m_3(k,\mathbf{q},1)\sin\frac{k+2q_1+4q_2}{6}\sin\frac{k+2q_1-2q_2}{6},\\
&B_6^{(1)}(k,\mathbf{q}) = 2m_3(k,\mathbf{q},1)m_5(k,\mathbf{q},1)\sin\frac{k+2q_1+4q_2}{6}\sin\frac{k+2q_1-2q_2}{6},\\
&B_1^{(2)}(k,\mathbf{q}) = 2\mathrm{i}\sin(q_1+q_2)\sin\frac{k+2q_1+4q_2}{6}\sin\frac{k+2q_1-2q_2}{6}m_6(k,\mathbf{q},1),\\
&B_2^{(2)}(k,\mathbf{q}) = -\mathrm{i}m_1(k,\mathbf{q},1)m_6(k,\mathbf{q},1)\sin(q_1+q_2),\\
&B_5^{(2)}(k,\mathbf{q}) = -\mathrm{i}m_2(k,\mathbf{q},1)m_3(k,\mathbf{q},1)\sin q_2,\\
&B_6^{(2)}(k,\mathbf{q}) = 2\mathrm{i}m_3(k,\mathbf{q},1)\sin q_2\sin\frac{k+2q_1+4q_2}{6}\sin\frac{k-4q_1-2q_2}{6},\\
&B_1^{(3)}(k,\mathbf{q}) = m_6(k,\mathbf{q},1)\sin q_2\sin(q_1+q_2),\\
&B_6^{(3)}(k,\mathbf{q}) = m_3(k,\mathbf{q},1)\sin q_2\sin(q_1+q_2),\\
&B_l^{(j)}(k,\mathbf{q}) = 0, \qquad (j,l) = (2,3-4),(3,2-5),
\end{aligned}
\tag{B.2}
$$

and its dual.

For $\Delta_0 = \Delta_2 = 0$, $\Delta_1 = 1$, the space of solutions additional to (95) and (101) is generated by the vector

$$
\begin{aligned}
&B_1^{(1)}(k,\mathbf{q}) = B_2^{(1)}(k,\mathbf{q}) = m_5(k,\mathbf{q},1)m_6(k,\mathbf{q},1),\\
&B_3^{(1)}(k,\mathbf{q}) = B_4^{(1)}(k,\mathbf{q}) = -2m_6(k,\mathbf{q},1)\sin\frac{k+2q_1+4q_2}{6}\sin\frac{k-4q_1-2q_2}{6},\\
&B_5^{(1)}(k,\mathbf{q}) = B_6^{(1)}(k,\mathbf{q}) = -2m_5(k,\mathbf{q},1)\sin\frac{k+2q_1-2q_2}{6}\sin\frac{k-4q_1-2q_2}{6},\\
&B_3^{(2)}(k,\mathbf{q}) = \mathrm{i}m_6(k,\mathbf{q},1)\sin(q_1+q_2),\\
&B_4^{(2)}(k,\mathbf{q}) = -2\mathrm{i}\sin(q_1+q_2)\sin\frac{k+2q_1-2q_2}{6}\sin\frac{k-4q_1-2q_2}{6}\\
&B_5^{(2)}(k,\mathbf{q}) = -2\mathrm{i}\sin q_1\sin\frac{k+2q_1+4q_2}{6}\sin\frac{k-4q_1-2q_2}{6},\\
&B_6^{(2)}(k,\mathbf{q}) = \mathrm{i}m_5(k,\mathbf{q},1)\sin q_1,\\
&B_4^{(3)}(k,\mathbf{q}) = B_5^{(3)}(k,\mathbf{q}) = -\sin q_1\sin(q_1+q_2),\\
&B_l^{(j)}(k,\mathbf{q}) = 0, \qquad (j,l) = (2,1-2),(3,1-3),(3,6),
\end{aligned}
\tag{B.3}
$$

and its dual.

For $\Delta_1 = \Delta_2 = 3/2$, $\Delta_0 = 0$, the space of solutions additional to (95) and (102) is generated by the vector

$$
\begin{aligned}
&B_2^{(1)}(k,\mathbf{q}) = -2m_5(k,\mathbf{q},3/2)\sin q_1\sin q_2,\\
&B_3^{(1)}(k,\mathbf{q}) = -2m_2(k,\mathbf{q},3/2)\sin q_1\sin q_2,\\
&B_1^{(2)}(k,\mathbf{q}) = m_4(k,\mathbf{q},3/2)m_5(k,\mathbf{q},3/2)m_6(k,\mathbf{q},3/2),\\
&B_2^{(2)}(k,\mathbf{q}) = m_1(k,\mathbf{q},3/2)m_5(k,\mathbf{q},3/2)m_6(k,\mathbf{q},3/2),\\
&B_3^{(2)}(k,\mathbf{q}) = m_1(k,\mathbf{q},3/2)m_2(k,\mathbf{q},3/2)m_6(k,\mathbf{q},3/2),\\
&B_4^{(2)}(k,\mathbf{q}) = m_1(k,\mathbf{q},3/2)m_2(k,\mathbf{q},3/2)m_3(k,\mathbf{q},3/2),
\end{aligned}
$$

$$
\begin{aligned}
&B_5^{(2)}(k,\mathbf{q}) = -B_5^{(3)}(k,\mathbf{q}) = m_2(k,\mathbf{q},3/2)m_3(k,\mathbf{q},3/2)m_4(k,\mathbf{q},3/2),\\
&B_6^{(2)}(k,\mathbf{q}) = -B_6^{(3)}(k,\mathbf{q}) = m_3(k,\mathbf{q},3/2)m_4(k,\mathbf{q},3/2)m_5(k,\mathbf{q},3/2),\\
&B_1^{(3)}(k,\mathbf{q}) = -m_4(k,\mathbf{q},3/2)m_5(k,\mathbf{q},3/2)\left[\cos\left(\frac{k-q_1-2q_2}{3}\right)-\cos q_1-\frac{\mathrm{e}^{\mathrm{i}q_1}}{2}\right],\\
&B_2^{(3)}(k,\mathbf{q}) = -m_1(k,\mathbf{q},3/2)m_5(k,\mathbf{q},3/2)\left[\cos\left(\frac{k-q_1-2q_2}{3}\right)-\cos q_1-\frac{\mathrm{e}^{\mathrm{i}q_1}}{2}\right],\\
&B_3^{(3)}(k,\mathbf{q}) = -m_2(k,\mathbf{q},3/2)m_6(k,\mathbf{q},3/2)\left[\cos\left(\frac{k+2q_1+q_2}{3}\right)-\cos q_2-\frac{\mathrm{e}^{-\mathrm{i}q_2}}{2}\right],\\
&B_4^{(3)}(k,\mathbf{q}) = -m_2(k,\mathbf{q},3/2)m_3(k,\mathbf{q},3/2)\left[\cos\left(\frac{k+2q_1+q_2}{3}\right)-\cos q_2-\frac{\mathrm{e}^{-\mathrm{i}q_2}}{2}\right],\\
&B_j(k,\mathbf{q}) = 0, \qquad (j,l) = (1,1),(1,4-6),
\end{aligned}
\tag{B.4}
$$

and its dual.

For $\Delta_0 = -3/2$, $\Delta_1 = \Delta_2 = 0$, the space of solutions additional to (95) and (102) is generated by the vector

$$
\begin{aligned}
&B_1^{(1)}(k,\mathbf{q}) = B_6^{(1)}(k,\mathbf{q}) = -\mathrm{i}m_5(k,\mathbf{q},-3/2)m_6(k,\mathbf{q},-3/2)\sin q_2,\\
&B_4^{(1)}(k,\mathbf{q}) = B_5^{(1)}(k,\mathbf{q}) = \mathrm{i}m_1(k,\mathbf{q},-3/2)\sin q_1\\
&\qquad\cdot\left(\cos\frac{k+q_2-q_1}{3}+\cos(q_1+q_2)+\frac{\mathrm{e}^{-\mathrm{i}(q_1+q_2)}}{2}\right),\\
&B_l^{(2)}(k,\mathbf{q}) = m_1(k,\mathbf{q},-3/2)m_5(k,\mathbf{q},-3/2)m_6(k,\mathbf{q},-3/2), \qquad l=1,2,3,4,5,6,\\
&B_3^{(3)}(k,\mathbf{q}) = B_4^{(3)}(k,\mathbf{q}) = \mathrm{i}m_1(k,\mathbf{q},-3/2)m_6(k,\mathbf{q},-3/2)\sin(q_1+q_2),\\
&B_5^{(3)}(k,\mathbf{q}) = B_6^{(3)}(k,\mathbf{q}) = \mathrm{i}m_5(k,\mathbf{q},-3/2)\sin q_1\\
&\qquad\cdot\left(\cos\frac{k+2q_1+q_2}{3}+\cos q_2+\frac{\mathrm{e}^{\mathrm{i}q_2}}{2}\right),\\
&B_l^{(j)}(k,\mathbf{q}) = 0, \qquad (j,l) = (1,2-3),(3,1-2),
\end{aligned}
\tag{B.5}
$$

and its dual.

## Appendix C. $S = 2$ solutions

For $\Delta_1 = \pm 1$, $\Delta_2 = 0$, the space of solutions is spanned on

$$
\begin{aligned}
&C_1^{(1)}(k,\mathbf{q}) = C_6^{(1)}(k,\mathbf{q}) = -\mathrm{i}m_5(k,\mathbf{q})m_6(k,\mathbf{q},\pm1)\sin q_2,\\
&C_2^{(1)}(k,\mathbf{q}) = C_3^{(1)}(k,\mathbf{q}) = 0,\\
&C_4^{(1)}(k,\mathbf{q}) = C_5^{(1)}(k,\mathbf{q}) = \mathrm{i}m_1(k,\mathbf{q},\pm1)m_2(k,\mathbf{q},\pm1)\sin q_1,\\
&C_1^{(2)}(k,\mathbf{q}) = C_2^{(2)}(k,\mathbf{q}) = \pm m_1(k,\mathbf{q},\pm1)m_5(k,\mathbf{q},\pm1)m_6(k,\mathbf{q},\pm1),\\
&C_3^{(2)}(k,\mathbf{q}) = C_4^{(2)}(k,\mathbf{q}) = \pm m_1(k,\mathbf{q},\pm1)m_2(k,\mathbf{q},\pm1)m_6(k,\mathbf{q},\pm1),\\
&C_5^{(2)}(k,\mathbf{q}) = C_6^{(2)}(k,\mathbf{q}) = \pm m_2(k,\mathbf{q},\pm1)m_5(k,\mathbf{q},\pm1)(\mathrm{e}^{\mathrm{i}(k+2q_1-2q_2)/6}\mp\mathrm{e}^{-\mathrm{i}(k+2q_1-2q_2)/6}),
\end{aligned}
\tag{C.1}
$$

and its dual.

For $\Delta_1 = 0$, $\Delta_2 = \pm 1$, the space of solutions is spanned on

$$C_1^{(1)}(k, \mathbf{q}) = C_6^{(1)}(k, \mathbf{q}) = 0,$$
$$C_2^{(1)}(k, \mathbf{q}) = \mathrm{i} m_5(k, \mathbf{q}, \pm 1) m_6(k, \mathbf{q}, \pm 1) \sin q_2,$$
$$C_3^{(1)}(k, \mathbf{q}) = \mathrm{i} m_2(k, \mathbf{q}, \pm 1) m_6(k, \mathbf{q}, \pm 1) \sin q_2,$$
$$C_4^{(1)}(k, \mathbf{q}) = \mathrm{i} m_1(k, \mathbf{q}, \pm 1) m_6(k, \mathbf{q}, \pm 1) \sin(q_1 + q_2),$$
$$C_5^{(1)}(k, \mathbf{q}) = \mathrm{i} m_4(k, \mathbf{q}, \pm 1) m_6(k, \mathbf{q}, \pm 1) \sin(q_1 + q_2),$$
$$C_1^{(2)}(k, \mathbf{q}) = C_6^{(2)}(k, \mathbf{q}, \pm 1) = \mp m_4(k, \mathbf{q}, \pm 1) m_5(k, \mathbf{q}, \pm 1) m_6(k, \mathbf{q}, \pm 1),$$
$$C_2^{(2)}(k, \mathbf{q}) = \mp m_1(k, \mathbf{q}, \pm 1) m_5(k, \mathbf{q}, \pm 1) m_6(k, \mathbf{q}, \pm 1),$$
$$C_3^{(2)}(k, \mathbf{q}) = \mp m_6^2(k, \mathbf{q}, \pm 1)(\mathrm{e}^{\mathrm{i}(k+2q_1+4q_2)/6} \mp \mathrm{e}^{-\mathrm{i}(k+2q_1+4q_2)/6}),$$
$$C_4^{(2)}(k, \mathbf{q}) = \mp m_3(k, \mathbf{q}, \pm 1) m_6(k, \mathbf{q}, \pm 1)(\mathrm{e}^{\mathrm{i}(k+2q_1+4q_2)/6} \mp \mathrm{e}^{-\mathrm{i}(k+2q_1+4q_2)/6}),$$
$$C_5^{(2)}(k, \mathbf{q}) = \mp m_2(k, \mathbf{q}, \pm 1) m_4(k, \mathbf{q}, \pm 1) m_6(k, \mathbf{q}, \pm 1), \tag{C.2}$$

and its dual.

## References


[1] Dagotto E 1999 *Rep. Prog. Phys.* **62** 1525
[2] Schmidt K P and Uhrig G S 2005 *Mod. Phys. Lett.* B **19** 1179
[3] Batchelor M T, Guan X-W, Oelkers N and Tsuboi Z 2007 *Adv. Phys.* **56** 465
[4] Barnes T, Dagotto E, Riera J and Swanson E S 1993 *Phys. Rev.* B **47** 3196
[5] Brehmer S, Mikeska H-J, Müller M, Nagaosa N and Uchida S 1999 *Phys. Rev.* B **60** 329–34
[6] Kolezhuk A K and Mikeska H-J 1998 *Int. J. Mod. Phys.* B **12** 2325
[7] Bibikov P N 2005 *Phys. Rev.* B **72** 012416
[8] Bibikov P N 2006 *Phys. Rev.* B **73** 132402
[9] Bibikov P N and Vyazovsky M I 2007 *Phys. Rev.* B **75** 094420
[10] Bibikov P N 2007 *Phys. Rev.* B **76** 174431
[11] Essler F H L 2000 *Phys. Rev.* B **62** 3264
[12] Kirillov A N and Smirnov F A 1998 *Int. J. Mod. Phys.* A **3** 731
[13] Gaudin M 1983 *La Fonction D'onde de Bethe* (Paris: Masson)
[14] Babbit D and Gutkin E 1990 *Lett. Math. Phys.* **20** 91
[15] Faddeev L D 1998 How algebraic Bethe ansatz works for integrable models *Quantum Symmetries/Symmetries Quantique (Proc. Les Houches Summer School Session LXIV)* ed A Connes, K Gawedzki and J Zinn-Justin (Amsterdam: North-Holland)
[16] Korepin V E, Izergin A G and Bogoliubov N M 1993 *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge: Cambridge University Press)
[17] Kulish P P and Sklyanin E K 1982 *Quantum Spectral Transform Method. Recent Developments (Lecture Notes in Phys.) (Proc. Symp. on Integrable Quantum Fields)* ed J Hietarinta and C Montonen (New York: Springer) p 151
[18] Hess C 2008 *Eur. Phys. J. Special Top.* **151** 73–83
[19] Mütter K-H and Schmitt A 1995 *J. Phys. A: Math. Gen.* **28** 2265
[20] Bibikov P N 2003 *Phys. Lett.* A **314** 209
[21] Bibikov P N 2003 *Zap. Nauchn. Semin. POMI* **291** 24
[22] Bibikov P N 2007 *J. Phys. A: Math. Theor.* **40** 4683
[23] Gritsev V and Baeriswyl D 2003 *J. Phys. A: Math. Gen.* **36** 12129–42
[24] Sachdev S and Bhatt R N 1990 *Phys. Rev.* B **41** 9323
[25] Kotov V N, Sushkov O, Weihong Z and Oitmaa J 1998 *Phys. Rev. Lett.* **80** 5790–93

[26] Hikihara T, Momoi T and Hu X 2003 *Phys. Rev. Lett.* **90** 087204
[27] Momoi T, Hikihara T, Nakamura N and Hu X 2003 *Phys. Rev.* B **67** 174410
[28] Legeza Ö, Fáth G and Sólyom J 1997 *Phys. Rev.* B **55** 291–8
[29] Albeverio S, Fei S-M and Wang Y 1999 *Europhys. Lett.* **47** 364–70
[30] Wang Y 1999 *Phys. Rev.* B **60** 9236–9
[31] Kulish P P 2003 *J. Phys. A: Math. Gen.* **36** L489–93